SOME NEW EXAMPLES WITH QUASI-POSITIVE CURVATURE

M. KERIN

Abstract. As a means to better understanding manifolds with positive curvature, there has been much recent interest in the study of non-negatively curved manifolds which contain a point at which all 2-planes have positive curvature. We show that there are generalisations of the well-known Eschenburg spaces together with quotients of $S^7 \times S^7$ which admit metrics with this property.

It is an unfortunate fact that for a simply connected manifold which admits a metric of non-negative curvature there are no known obstructions to admitting positive curvature. While there exist many examples of manifolds with non-negative curvature, the known examples with positive curvature are very sparse (see [Zi] for a comprehensive survey of both situations). Other than the rank-one symmetric spaces there are isolated examples in dimensions $6, 7, 12, 13$ and 24 due to Wallach [Wa] and Berger [Ber], and two infinite families, one in dimension 7 (Eschenburg spaces; see [AW], [E1], [E2]) and the other in dimension 13 (Bazaikin spaces; see [Ba]). In recent developments, two distinct metrics with positive curvature on a particular cohomogeneity-one manifold have been proposed ([GVZ], [D]), while in [PW2] the authors propose that the Gromoll-Meyer exotic 7-sphere admits positive curvature, which would be the first exotic sphere known to exhibit this property.

In this paper we are interested in the study of manifolds which lie "between" those with non-negative and those with positive sectional curvature. It is hoped that the study of such manifolds will yield a better understanding of the differences between these two classes.

Recall that a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is said to have *quasi-positive curvature* (resp. *almost positive curvature*) if $(M, \langle\ ,\ \rangle)$ has non-negative sectional curvature and there is a point (resp. an open dense set of points) at which all 2-planes have positive sectional curvature.

Our main result is:

Theorem A.

(i) *Let $L_{p,q} \subset U(n+1) \times U(n+1)$, $n \geq 2$, be defined by*

$$L_{p,q} = \{(\mathrm{diag}(z^{p_1}, \dots, z^{p_{n+1}}), \mathrm{diag}(z^{q_1}, z^{q_2}, A)) \mid z \in S^1, A \in U(n-1)\},$$

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where $p_1, \ldots, p_{n+1}, q_1, q_2 \in \mathbb{Z}$.
If the action is free then the biquotient $E^{4n-1}_{p,q} = U(n+1)/\!\!/L_{p,q}$ *admits a metric with quasi-positive curvature whenever there exists* $1 \leq i < j \leq n+1$ *such that*

$$p_i \neq p_j \quad \text{and} \quad p_i + p_j \notin \{2q_1, 2q_2, q_1 + q_2\}.$$

(ii) *There exists a free circle action on* $S^7 \times S^7$ *such that the quotient* $M^{13} = S^1 \backslash (S^7 \times S^7)$ *admits a metric with quasi-positive curvature. Furthermore,* M^{13} *and* $\mathbb{C}P^3 \times S^7$ *has the same integral cohomology but are not homeomorphic.*

(iii) *There exists a free* S^3*-action on* $S^7 \times S^7$ *such that the quotient* $N^{11} = S^3 \backslash (S^7 \times S^7)$ *admits a metric with quasi-positive curvature. Furthermore,* N^{11} *and* $S^4 \times S^7$ *have the same integral cohomology but are not homeomorphic.*

One of the original motivations for studying manifolds with quasi-positive curvature was the Deformation Conjecture, which stated that if $(M, \langle\ ,\ \rangle)$ is a complete Riemannian manifold with quasi-positive curvature, then M admits a metric with positive curvature. Wilking [Wi] provided counter-examples when he showed that there are odd-dimensional, non-orientable manifolds which admit almost positive curvature. By Synge's Theorem such manifolds cannot admit positive curvature. However, all of Wilking's counter-examples have non-trivial fundamental group. Therefore it is still possible that the Deformation Conjecture holds for simply connected manifolds. Moreover, in [PW1] the authors ask whether a weaker version of the Deformation Conjecture is true, namely whether quasi-positive curvature be deformed to almost positive curvature.

Other than the Gromoll-Meyer exotic 7-sphere ([GM], [W], [EK], [PW2]), the only other previously known examples of manifolds with almost positive or quasi-positive curvature are given in [PW1], [Wi], [Ta1], and [Ke2].

In addition to his other examples, Wilking [Wi] has also shown that the homogeneous spaces $M^{4n-1}_{k,\ell} = U(n+1)/H_{k,\ell}$ admit metrics with almost positive curvature, where $k, \ell \in \mathbb{Z}$, $k\ell < 0$, $n \geq 2$, and

$$H_{k,\ell} = \{\operatorname{diag}(z^k, z^\ell, A) \mid z \in S^1, A \in U(n-1)\},$$

Tapp [Ta1] subsequently showed that all $M^{4n-1}_{k,\ell} = U(n+1)/H_{k,\ell}$ with $k, \ell \in \mathbb{Z}$, $(k, \ell) \neq (0, 0)$, admit quasi-positive curvature.

Furthermore, with these examples Wilking has shown that there are infinitely many homotopy types of simply connected manifolds within each dimension $4n - 1$ which admit almost positive curvature. When $n = 2$ the homogeneous spaces described by $M^{4n-1}_{k,\ell}$ are the 7-dimensional Aloff-Wallach spaces, $W^7_{k,\ell}$, [AW]. Recall that the Aloff-Wallach space $W^7_{k,\ell}$ admits a homogeneous metric with positive curvature if and only if $k\ell(k+\ell) \neq 0$. There is thus a unique Aloff-Wallach space, namely $W^7_{-1,1}$, which admits almost positive curvature but is not known to admit positive curvature.

The biquotients $E^{4n-1}_{p,q}$ in Theorem A(i) should be thought of as generalisations of the Eschenburg spaces, which arise when $n = 2$. In [E1] it is shown that infinitely many Eschenburg spaces admit positive curvature, while in [Ke2] it is shown that all Eschenburg spaces admit a metric with quasi-positive curvature. From our previous remarks on Wilking's work we see that there are infinitely many homotopy types of generalised Eschenburg spaces $E^{4n-1}_{p,q}$ for a fixed dimension $4n - 1$.

The paper is organised as follows. In Section 1 we review some notation and geometric techniques for biquotients. In Section 2 we review some facts about the Cayley numbers and the exceptional Lie group G_2. In Section 3 we describe the manifolds M^{13} and N^{11} of Theorem A as biquotients. We prove the curvature statements of Theorem A(ii) and (iii) in Section 4, while proof of the topological statements is postponed until Section 6. Section 5 is devoted to establishing Theorem A(i).

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1. Biquotient actions and metrics

In his Habilitation, [E1, '84], Eschenburg studied biquotients in great detail. The following section provides a review of the material in [E1] and establishes the basic language, notation and results which will be used throughout the remainder of the paper.

Let G be a compact Lie group, $U \subset G \times G$ a closed subgroup, and let U act on G via

$$(u_1, u_2) \star g = u_1 g u_2^{-1}, \quad g \in G, (u_1, u_2) \in U.$$

The action is free if and only if, for all non-trivial $(u_1, u_2) \in U$, u_1 is never conjugate to u_2 in G. The resulting manifold is called a *biquotient.*

Let $K \subset G$ be a closed subgroup, $\langle\ ,\ \rangle$ be a left-invariant, right K-invariant metric on G, and $U \subset G \times K \subset G \times G$ act freely on G as above. Let $g \in G$. Define

$$U^g_L := \{(g u_1 g^{-1}, u_2) \mid (u_1, u_2) \in U\}.$$

Since U acts freely on G, so too does U^g_L, and $G/\!\!/U$ is isometric to $G/\!\!/U^g_L$. This follows from the fact that left-translation $L_g : G \longrightarrow G$ is an isometry which satisfies $g u_1 g^{-1}(L_g g')u_2^{-1} = L_g(u_1 g' u_2^{-1})$. Therefore L_g induces an isometry of the orbit spaces $G/\!\!/U$ and $G/\!\!/U^g_L$.

Consider a Riemannian submersion $\pi : M^n \longrightarrow N^{n-k}$. By O'Neill's formula for Riemannian submersions, π is curvature non-decreasing. Therefore $\sec_M \geq 0$ implies $\sec_N \geq 0$, and zero-curvature planes on N lift to horizontal zero-curvature planes on M. Because of the Lie bracket term in the O'Neill

formula the converse is not true in general, namely horizontal zero-curvature planes in M cannot be expected to project to zero-curvature planes on N.

Let $K \subset G$ be Lie groups, $\mathfrak{k} \subset \mathfrak{g}$ the corresponding Lie algebras, and $\langle\ ,\ \rangle$ a non-negatively curved left-invariant metric on G which is right-invariant under K. We can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\langle\ ,\ \rangle$. Given $X \in \mathfrak{g}$ we will always use $X_\mathfrak{k}$ and $X_\mathfrak{p}$ to denote the $\mathfrak{k}$ and $\mathfrak{p}$ components of X respectively.

Recall that

$$G \cong (G \times K)/\Delta K$$

via $(g,k) \longmapsto gk^{-1}$, where ΔK is the free, isometric, diagonal action of K on the right of $G \times K$. Notice that the restriction of $\langle\ ,\ \rangle$ to K is bi-invariant. Thus we may define a new left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ (with $\sec \geq 0$) on G via the Riemannian submersion

$$\begin{array}{rcl} (G \times K, \langle\ ,\ \rangle \oplus t\langle\ ,\ \rangle|_\mathfrak{k}) & \longrightarrow & (G, \langle\ ,\ \rangle_1) \\ (g,k) & \longmapsto & gk^{-1}, \end{array}$$

where $t > 0$ and

$$\langle X, Y\rangle_1 = \langle X, \Phi(Y)\rangle, \tag{1.1}$$

where $\Phi(Y) = Y_\mathfrak{p} + \lambda Y_\mathfrak{k}$, $\lambda = \frac{t}{t+1} \in (0,1)$. Furthermore, it is clear that the metric tensor Φ is invertible with inverse described by $\Phi^{-1}(Y) = Y_\mathfrak{p} + \frac{1}{\lambda} Y_\mathfrak{k}$.

Suppose $\sigma = \mathrm{Span}\,\{\Phi^{-1}(X), \Phi^{-1}(Y)\} \subset \mathfrak{g}$ is a zero-curvature plane with respect to the metric $\langle\ ,\ \rangle_1$, i.e. $\sec_1(\sigma) = 0$. By the O'Neill formula σ must therefore lift to a horizontal zero-curvature plane $\widetilde{\sigma} \subset \mathfrak{g} \oplus \mathfrak{k}$ with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. It is easy to check that the horizontal lift of a vector $\Phi^{-1}(X) \in \mathfrak{g}$ to $\mathfrak{g} \oplus \mathfrak{k}$ is given by $\left(X, -\frac{1}{t} X_\mathfrak{k}\right)$. Then clearly

$$\widetilde{\sigma} = \mathrm{Span}\,\left\{ \left(X, -\frac{1}{t} X_\mathfrak{k}\right), \left(Y, -\frac{1}{t} Y_\mathfrak{k}\right) \right\}.$$

But, since $\langle\!\langle\ ,\ \rangle\!\rangle$ is a non-negatively curved product metric, it follows immediately by considering the unnormalised curvature that $\widetilde{\sigma}$ has zero-curvature if and only if $[X_\mathfrak{k}, Y_\mathfrak{k}] = 0$ and the plane $\mathrm{Span}\,\{X, Y\} \subset \mathfrak{g}$ has zero-curvature with respect to the original metric $\langle\ ,\ \rangle$, i.e. $\sec(X,Y) = 0$.

From [Ta2], which generalizes similar results in [E1] and [Wi], we know that if $\langle\ ,\ \rangle$ is induced by a Riemannian submersion to G from a bi-invariant metric on some Lie group L, then in fact $\sec_1(\sigma) = 0$ if and only if $\sec(\widetilde{\sigma}) = 0$ with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$, i.e. if and only if $\sec(X,Y) = 0$ and $[X_\mathfrak{k}, Y_\mathfrak{k}] = 0$. We will always be in this situation as throughout the paper we will use only the metrics described in Examples (a) and (b) below.

Example (a). Suppose that (G,K) is a symmetric pair and that the initial metric $\langle\ ,\ \rangle = \langle\ ,\ \rangle_0$ is a bi-invariant metric on G. As in (1.1), equip G with a new metric

$$\langle X, Y\rangle_1 = \langle X, \Phi_1(Y)\rangle_0, \tag{1.2}$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\langle\ ,\ \rangle_0$ and $\Phi_1(Y) = Y_\mathfrak{p} + \lambda_1 Y_\mathfrak{k}$. Then $\sigma = \mathrm{Span}\,\{\Phi_1^{-1}(X), \Phi_1^{-1}(Y)\} \subset \mathfrak{g}$ has zero-curvature with respect to $\langle\ ,\ \rangle_1$,

i.e. $\sec_1(\sigma) = 0$, if and only if

$$0 = [X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}]. \tag{1.3}$$

The proof of this follows immediately from our previous discussion together with the fact that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ whenever (G, K) is a symmetric pair.

Example (b). Let $G \supset K \supset H$ be a chain of subgroups and suppose that both (G, K) and (K, H) are symmetric pairs. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ be the corresponding orthogonal decompositions with respect to the bi-invariant metric $\langle\ ,\ \rangle_0$ on G. Start with the metric $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1$ defined by Example (a). Now define the metric $\langle\ ,\ \rangle_2$ on G as in (1.1), where K is replaced by H, $s > 0$ takes the role of t, and Ψ replaces Φ:

$$\begin{aligned} \langle X, Y\rangle_2 &= \langle X, \Psi(Y)\rangle_1 && (1.4)\\ &= \langle X, \Phi_2(Y)\rangle_0 && (1.5) \end{aligned}$$

with $\Phi_2(Y) = Y_{\mathfrak{p}} + \lambda_1 Y_{\mathfrak{m}} + \lambda_1\lambda_2 Y_{\mathfrak{h}}$, $\lambda_2 = \frac{s}{s+1}$, and $\Psi(Y) = \Phi_1^{-1}\Phi_2(Y) = Y_{\mathfrak{p}} + Y_{\mathfrak{m}} + \lambda_2 Y_{\mathfrak{h}}$.

Let $\sigma = \mathrm{Span}\,\{\Psi^{-1}(X), \Psi^{-1}(Y)\} \subset \mathfrak{g}$. Then, by our discussion prior to Example (a), $\sec_2(\sigma) = 0$ if and only if $\sec_1(X, Y) = 0$ and $[X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$. By again considering horizontal lifts it is not difficult to check that $\sec_1(X, Y) = 0$ if and only if conditions (1.3) hold as for $\sec_1(\Phi_1^{-1}(X), \Phi_1^{-1}(Y)) = 0$. Hence $\sec_2(\sigma) = 0$ if and only if

$$0 = [X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{m}}, Y_{\mathfrak{m}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}], \tag{1.6}$$

where we have used the fact that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ since (K, H) is a symmetric pair.

Now that we have described how to induce new metrics on G from old ones and derived zero-curvature conditions for these metrics, we proceed to consider biquotients $G/\!\!/U$. Let $\Delta G = \{(g, g) \mid g \in G\}$. Then, if the two-sided action of U on G is free, $\Delta G \times U$ acts freely on $G \times G$ via

$$((g, g), (u_1, u_2)) \star (g_1, g_2) = (g g_1 u_1^{-1}, g g_2 u_2^{-1}), \tag{1.7}$$

with $(g, g) \in \Delta G$, $(u_1, u_2) \in U$, $(g_1, g_2) \in G \times G$, and there is a canonical diffeomorphism

$$\Delta G\backslash(G \times G)/U \cong G/\!\!/U \tag{1.8}$$

induced by the map

$$\begin{aligned} G \times G &\longrightarrow G\\ (g_1, g_2) &\longmapsto g_1^{-1} g_2. \end{aligned}$$

Let K_1 and K_2 be arbitrary subgroups of G. We define left-invariant metrics, $\langle\ ,\ \rangle_{K_1}$ and $\langle\ ,\ \rangle_{K_2}$, on G as in (1.1). Equip $G \times G$ with a left-invariant, right $(K_1 \times K_2)$-invariant product metric $\langle\!\langle\ ,\ \rangle\!\rangle = \langle\ ,\ \rangle_{K_1} \oplus \langle\ ,\ \rangle_{K_2}$. If $U \subset K_1 \times K_2$ then the $\Delta G \times U$ action is by isometries and $\langle\!\langle\ ,\ \rangle\!\rangle$ induces a metric on $G/\!\!/U$. Our goal is to determine when a plane tangent to $G/\!\!/U$ has zero-curvature with respect to this induced metric.

By (1.8) and our choice of metric, the quotient map $(G \times G, \langle\!\langle\ ,\ \rangle\!\rangle) \longrightarrow G/\!\!/U$ is a Riemannian submersion. O'Neill's formula implies that a zero-curvature plane tangent to $G/\!\!/U$ must lift to a horizontal zero-curvature plane with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. As in the case of metrics on G, if $\langle\!\langle\ ,\ \rangle\!\rangle$ is induced from a bi-invariant metric on some Lie group L, then [Ta2] implies that horizontal zero-curvature planes with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ must project to zero-curvature planes in $G/\!\!/U$. For our purposes this will always be true since we will consider only metrics as in Examples (a) and (b).

We must determine what it means for a plane to be horizontal with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ and the $\Delta G \times U$ action. Since each $\Delta G \times U$ orbit passes through some point of the form $(g, e) \in G \times G$, where e is the identity element of G, we may restrict our attention to such points.

Recall that $\langle\!\langle\ ,\ \rangle\!\rangle$ is left-invariant. Therefore, letting $\mathfrak{u}$ denote the Lie algebra of U, the vertical subspace at $(g, e) \in G \times G$ is given by

$$\mathcal{V}_g = \left\{ \left(\mathrm{Ad}_{g^{-1}} X - Y_1, X - Y_2\right) \mid X \in \mathfrak{g}, (Y_1, Y_2) \in \mathfrak{u} \right\}$$

after left-translation to $(e, e) \in G \times G$. Note that this is independent of the choice of left-invariant metric on $G \times G$.

Thus, with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$, the horizontal subspace at (g, e) is

$$\mathcal{H}_g = \left\{ \left(\Omega_1^{-1}(-\mathrm{Ad}_{g^{-1}} X), \Omega_2^{-1}(X)\right) \mid \langle X, \mathrm{Ad}_g Y_1 - Y_2 \rangle_0 = 0 \ \forall\ (Y_1, Y_2) \in \mathfrak{u} \right\} \tag{1.9}$$

where Ω_1 and Ω_2 are the metric tensors relating the left-invariant metrics $\langle\ ,\ \rangle_{K_1}$ and $\langle\ ,\ \rangle_{K_1}$ respectively to a fixed bi-invariant metric $\langle\ ,\ \rangle_0$ on G, i.e. $\langle X, Y \rangle_{K_i} = \langle X, \Omega_i(Y) \rangle_0$, $i = 1, 2$. We recall that the metric tensors in Examples (a) and (b) are given by Φ_1 and Φ_2 respectively, as shown in (1.2) and (1.5).

In particular, (1.9) shows that a horizontal 2-plane σ in $(G \times G, \langle\!\langle\ ,\ \rangle\!\rangle)$ must project to a 2-plane on each factor, denoted by $\check{\sigma}_1$ and $\check{\sigma}_2$ respectively. Moreover, since $\langle\!\langle\ ,\ \rangle\!\rangle$ is a product metric, $\sec(\sigma) = 0$ if and only if $\sec_1(\check{\sigma}_i) = 0$, $i = 1, 2$. Thus, for product metrics involving the metrics described by Examples (a) and (b), we may apply conditions (1.3) and (1.6) respectively in order to determine when a horizontal plane σ has zero-curvature.

2. The Cayley numbers, G_2 and its Lie algebra

We recall without proof some well known facts about Cayley numbers, the Lie group G_2 and its Lie algebra. More details may be found in [GWZ] and [M].

We may write the Cayley numbers as $Ca = \mathbb{H} + \mathbb{H}\ell$. Thus we have a natural orthonormal basis

$$\{e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = \ell, e_5 = i\ell, e_6 = j\ell, e_7 = k\ell\}$$

for Ca. Note that this description of Ca differs slightly from that given in [M], and accounts for the difference which occurs in the description of the

Lie algebra $\mathfrak{g}_2$ in Theorem 2.2. Multiplication in Ca is non-associative and defined via

$$(a+b\ell)(c+d\ell) = (ac - \bar{d}b) + (da + b\bar{c})\ell, \quad a,b,c,d \in \mathbb{H}. \tag{2.1}$$

Hence we have the following multiplication table, where the order of multiplication is given by (row)$*$(column):

	$e_1 = i$	$e_2 = j$	$e_3 = k$	$e_4 = \ell$	$e_5 = i\ell$	$e_6 = j\ell$	$e_7 = k\ell$
$e_1 = i$	-1	k	$-j$	$i\ell$	$-\ell$	$-k\ell$	$j\ell$
$e_2 = j$	$-k$	-1	i	$j\ell$	$k\ell$	$-\ell$	$-i\ell$
$e_3 = k$	j	$-i$	-1	$k\ell$	$-j\ell$	$i\ell$	$-\ell$
$e_4 = \ell$	$-i\ell$	$-j\ell$	$-k\ell$	-1	i	j	k
$e_5 = i\ell$	ℓ	$-k\ell$	$j\ell$	$-i$	-1	$-k$	j
$e_6 = j\ell$	$k\ell$	ℓ	$-i\ell$	$-j$	k	-1	$-i$
$e_7 = k\ell$	$-j\ell$	$i\ell$	ℓ	$-k$	$-j$	i	-1

TABLE 1. Multiplication table for Ca

Recall that the Lie group G_2 is the automorphism group of $Ca \cong \mathbb{R}^8$. In fact G_2 is a connected subgroup of $SO(7) \subset SO(8)$, where $SO(8)$ acts on $Ca \cong \mathbb{R}^8$ by orthogonal transformations and $SO(7)$ is that subgroup consisting of elements which leave $e_0 = 1$ fixed. $SO(8)$ also contains two copies of $Spin(7)$ which are not conjugate in $SO(8)$, and G_2 is the intersection of these two subgroups.

As our eventual goal is to prove Theorem A(ii) and (iii), it is useful to recall the fact that G_2 appears in the descriptions of some interesting homogeneous spaces. The following statements are well-known and follow from applications of the triality principle for $SO(8)$. More details may be found in, for example, [M], [J, p. 93].

Theorem 2.1.

(i) $Spin(7)/G_2 = S^7$, *which inherits positive curvature from the bi-invariant metric on* $Spin(7)$*;*
(ii) $Spin(8)/G_2 = S^7 \times S^7$ *and* $SO(8)/G_2 = (S^7 \times S^7)/\mathbb{Z}_2$*, where* $\mathbb{Z}_2 = \{\pm \mathrm{id}\}$*;*
(iii) $G_2/SU(3) = S^6$*.*

We now turn our attention to the Lie algebra of G_2. The proof of the following theorem follows exactly as in [M] except that we use the basis and multiplication conventions for Ca as in Table 1. Recall that $\mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) \mid A^t = -A\}$.

Theorem 2.2. *The Lie algebra of G_2, denoted by $\mathfrak{g}_2$, consists of matrices $A = (a_{ij}) \in \mathfrak{so}(7)$ which satisfy $a_{ij} + a_{ji} = 0$ and*

$$\begin{aligned} a_{23} + a_{45} + a_{76} &= 0 \\ a_{12} + a_{47} + a_{65} &= 0 \\ a_{13} + a_{64} + a_{75} &= 0 \\ a_{14} + a_{72} + a_{36} &= 0 \\ a_{15} + a_{26} + a_{37} &= 0 \\ a_{16} + a_{52} + a_{43} &= 0 \\ a_{17} + a_{24} + a_{53} &= 0. \end{aligned}$$

Hence $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ is 14-dimensional and consists of matrices of the form

$$\begin{pmatrix} 0 & x_1+x_2 & y_1+y_2 & x_3+x_4 & y_3+y_4 & x_5+x_6 & y_5+y_6 \\ -(x_1+x_2) & 0 & \alpha_1 & -y_5 & x_5 & -y_3 & x_3 \\ -(y_1+y_2) & -\alpha_1 & 0 & x_6 & y_6 & -x_4 & -y_4 \\ -(x_3+x_4) & y_5 & -x_6 & 0 & \alpha_2 & y_1 & -x_1 \\ -(y_3+y_4) & -x_5 & -y_6 & -\alpha_2 & 0 & x_2 & y_2 \\ -(x_5+x_6) & y_3 & x_4 & -y_1 & -x_2 & 0 & \alpha_1+\alpha_2 \\ -(y_5+y_6) & -x_3 & y_4 & x_1 & -y_2 & -(\alpha_1+\alpha_2) & 0 \end{pmatrix}. \tag{2.2}$$

Recall that G_2 is a rank 2 Lie group. Therefore an examination of the elements (2.2) of $\mathfrak{g}_2$ reveals that the maximal torus of G_2 is given by

$$T^2 = \left\{ \begin{pmatrix} 1 & & & \\ & R(\theta) & & \\ & & R(\varphi) & \\ & & & R(\theta+\varphi) \end{pmatrix} \;\middle|\; R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}. \tag{2.3}$$

3. Free isometric actions on $SO(8)$

Consider the rank one symmetric pair $(G, K) = (SO(8), SO(7))$ where

$$\begin{aligned} SO(7) &\hookrightarrow SO(8) \\ A &\longmapsto \begin{pmatrix} 1 & \\ & A \end{pmatrix}, \end{aligned}$$

with Lie algebras $\mathfrak{g}, \mathfrak{k}$ respectively. Let $\langle X, Y \rangle_0 = -\operatorname{tr}(XY)$ be a bi-invariant metric on G. With respect to $\langle\ ,\ \rangle_0$ we thus have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. As in (1.2) we define a left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ on G by

$$\langle X, Y \rangle_1 = \langle X, \Phi_1(Y) \rangle_0, \tag{3.1}$$

where $\Phi_1(Y) = Y_{\mathfrak{p}} + \lambda_1 Y_{\mathfrak{k}}$, $\lambda_1 \in (0, 1)$. Recall that from Example (a) we know that a plane

$$\sigma = \operatorname{Span}\{\Phi_1^{-1}(X), \Phi_1^{-1}(Y)\} \subset \mathfrak{g}$$

has zero-curvature with respect to $\langle\ ,\ \rangle_1$ if and only if

$$0 = [X, Y] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]. \tag{3.2}$$

We now equip $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$.

Consider an isometric action of $U := S^1 \times G_2 \subset K \times K$ on $SO(8)$ defined by

$$A \longmapsto \widetilde{R}(\theta) \cdot A \cdot g^{-1}, \tag{3.3}$$

where $A \in SO(8)$, $g \in G_2$, and

$$\widetilde{R}(\theta) = \begin{pmatrix} I_{2\times 2} & & & \\ & R(p_1\theta) & & \\ & & R(p_2\theta) & \\ & & & R(p_3\theta) \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.4}$$

From (1.8) we know that $\Delta G \backslash G \times G/U \cong G/\!\!/U$ whenever the biquotient action of U on G is free.

Lemma 3.1. *$\Delta G \times U$ acts freely and isometrically on $(G \times G, \langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1)$ if and only if (p_1, p_2, p_3) is equal to $(0, 0, 1)$ (up to sign and permutations of the p_i).*

Proof. Recall that conjugation of either factor of U by elements of G is a diffeomorphism, and that a biquotient action is free if and only if non-trivial elements in each factor are never conjugate to one another in G. Thus we need only show that non-trivial elements of S^1 and T^2 are never conjugate in G if and only if $(p_1, p_2, p_3) = (0, 0, 1)$ up to sign and permutations of the p_i, where T^2 is the maximal torus of G_2 described in (2.3). This amounts to investigating when the sets of 2×2 blocks on each side are equal up to conjugation by an element of the Weyl group of $SO(8)$. We recall that the Weyl group of $SO(2n)$ acts via permutations of the 2×2 blocks and changing an even number of signs, where by a change of sign we mean $R(\theta) \longmapsto R(-\theta)$. A simple calculation then yields the result. $\square$

Note that there are many other free $S^1 \times G_2$ actions on G. For example, there is a free S^1 action on the left of G/G_2 by matrices of the form

$$\begin{pmatrix} R(\theta) & & & \\ & R(\theta) & & \\ & & R(\theta) & \\ & & & R(k\theta) \end{pmatrix} \tag{3.5}$$

where $(k, 3) = 1$. However, it is clear that only the action in Lemma 3.1 is isometric with respect to the metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$ on $G \times G$.

It follows immediately from the long exact homotopy sequence for fibrations that a biquotient $Spin(8)/\!\!/(S^1 \times G_2) = S^1 \backslash (S^7 \times S^7)$ must be simply connected. By the lifting criterion for covering spaces the action by U on $SO(8)$ described above lifts to some action by $S^1 \times G_2$ on $Spin(8)$. Therefore, together with Theorem 2.1, one might expect that the resulting simply

connected biquotient $Spin(8)/\!\!/(S^1 \times G_2) = S^1\backslash(S^7 \times S^7)$ is a non-trivial finite cover of $SO(8)/\!\!/(S^1 \times G_2)$. In fact the lemma below will demonstrate that this covering map is a diffeomorphism.

Lemma 3.2. *$M^{13} := SO(8)/\!\!/(S^1 \times G_2)$ is simply connected and hence a quotient of $S^7 \times S^7$ by an S^1 action.*

Proof. Consider a general embedding

$$\begin{array}{ccc} S^1_q & \hookrightarrow & SO(8) \\ R(\theta) & \longmapsto & \begin{pmatrix} R(q_1\theta) & & & \\ & R(q_2\theta) & & \\ & & R(q_3\theta) & \\ & & & R(q_4\theta) \end{pmatrix} \end{array}$$

where $q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}^4$, where $R(u) \in SO(2)$. The long exact homotopy sequence for the fibration $S^1_q \times G_2 \longrightarrow SO(8) \longrightarrow SO(8)/\!\!/S^1_q \times G_2$ yields

$$\ldots \longrightarrow \pi_1(S^1_q \times G_2) = \mathbb{Z} \longrightarrow \pi_1(SO(8)) = \mathbb{Z}_2 \longrightarrow \pi_1(SO(8)/\!\!/S^1_q \times G_2) \longrightarrow 0.$$

Thus to obtain the desired result we need only show that the map $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ is surjective.

Recall that the homomorphism $\iota_* : \pi_1(S^1_q) \longrightarrow \pi_1(SO(n))$ is determined by the weights $q = (q_1, \ldots, q_m)$, $m = \lfloor \frac{n}{2} \rfloor$, of the embedding, namely $\iota_*(1) = \sum q_i \mod 2$. Therefore ι_* is onto exactly when $\sum q_i$ is odd. In our case we have $q = (0, 0, 0, 1)$, and so ι_* is a surjection. $\square$

Notice that the action of U on $SO(8)$ given in Lemma 3.1 may be enlarged to an isometric action by $SO(3) \times G_2$, and the resulting biquotient we call N^{11}. Now recall that for all n we have a 2-fold cover $Spin(n) \longrightarrow SO(n)$ with $\pi_1(Spin(n)) = 0$ and $\pi_1(SO(n)) = \mathbb{Z}_2$. Thus, by the lifting criterion for covering spaces, the inclusion $SO(3) \hookrightarrow SO(8)$ must lift to $Spin(3) = S^3 \hookrightarrow Spin(8)$. As in the case of $U = S^1 \times G_2$ above we show that $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ is simply connected and hence diffeomorphic to $Spin(8)/\!\!/S^3 \times G_2 = S^3\backslash(S^7 \times S^7)$.

Lemma 3.3. *$N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ is simply connected and hence a quotient of $S^7 \times S^7$ by an S^3 action.*

Proof. Consider the chain of embeddings $i \circ j : S^1 = SO(2) \hookrightarrow SO(3) \hookrightarrow SO(8)$ given by enlarging S^1 above to an $SO(3)$ in $SO(8)$. We thus have an induced homomorphism on fundamental groups $(i \circ j)_* = i_* \circ j_* : \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$. But i_* and $(i \circ j)_*$ are simply the homomorphism ι_* from Lemma 3.2. Hence $i_*(1) = 1 \mod 2$ and $(i \circ j)_*(1) = 1 \mod 2$. This implies $j_*(1) = 1 \mod 2$ and therefore j_* is a surjection. An examination of the long exact homotopy sequence of the fibration $SO(3) \times G_2 \longrightarrow SO(8) \longrightarrow N^{11}$ yields the result. $\square$

4. Quasi-positive curvature of M^{13} and N^{11}

Given Lemma 3.2 we are now in a position to perform the curvature computations for the circle quotient of $S^7 \times S^7$ mentioned in Theorem A, namely

$$M^{13} = SO(8)/\!\!/(S^1 \times G_2) = G/\!\!/U,$$

where S^1 is the circle giving a free isometric action U as in Lemma 3.1.

Consider the inclusions $G = SO(8) \supset K = SO(7) \supset G_2$. With respect to the bi-invariant metric $\langle X, Y\rangle_0 = -\operatorname{tr}(XY)$ on G we have

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{m} \oplus \mathfrak{g}_2,$$

where

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & w \\ \hline -w^t & 0 \end{array} \right) \;\middle|\; w \in \mathbb{R}^7 \right\} \tag{4.1}$$

and, by (2.2),

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 0 & -v_1 & 0 & v_7 & v_6 & -v_5 & v_4 & -v_3 \\ 0 & -v_2 & -v_7 & 0 & -v_5 & -v_6 & v_3 & v_4 \\ 0 & -v_3 & -v_6 & v_5 & 0 & v_7 & -v_2 & v_1 \\ 0 & -v_4 & v_5 & v_6 & -v_7 & 0 & -v_1 & -v_2 \\ 0 & -v_5 & -v_4 & -v_3 & v_2 & v_1 & 0 & -v_7 \\ 0 & -v_6 & v_3 & -v_4 & -v_1 & v_2 & v_7 & 0 \end{array} \right) \;\middle|\; v_i \in \mathbb{R} \right\}. \tag{4.2}$$

Let $U = S^1 \times G_2 \subset K \times K$ be as in Lemma 3.1. Thus, equipping $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$ as before, we may induce a metric on $G/\!\!/U$ via the diffeomorphism

$$\Delta G \backslash G \times G/U \longrightarrow G/\!\!/U.$$

As discussed in Section 1, we may restrict our attention to points of the form $(A, I) \in G \times G$. Let $\mathfrak{s} \subset \mathfrak{g}$ denote the Lie algebra of the S^1 factor of U. By (1.9) the horizontal subspace at (A, I) with respect to $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$ is given by

$$\mathcal{H}_A = \{(-\Phi_1^{-1}(\operatorname{Ad}_{A^t} W), \Phi_1^{-1}(W)) \mid W_{\mathfrak{g}_2} = 0, \langle W, \operatorname{Ad}_A \Theta\rangle_0 = 0, \forall\ \Theta \in \mathfrak{s}\}.$$

Suppose that $\sigma = \subset \mathfrak{g} \oplus \mathfrak{g}$ is a horizontal zero-curvature plane at $(A, I) \in G \times G$. Since we have equipped $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_1$, our discussion in Section 1 shows that σ must project to zero-curvature planes $\check{\sigma}_i$, $i = 1, 2$, on each factor.

Lemma 4.1. *Suppose*

$$\sigma = \operatorname{Span}\{(-\Phi_1^{-1}(\operatorname{Ad}_{A^t} X), \Phi_1^{-1}(X)), (-\Phi_1^{-1}(\operatorname{Ad}_{A^t} Y), \Phi_1^{-1}(Y))\}$$

is a horizontal zero-curvature plane at $(A, I) \in G \times G$. Then it may be assumed without loss of generality that $X \in \mathfrak{p}$ and $Y \in \mathfrak{m}$.

Proof. As $\check{\sigma}_2$ is a zero-curvature plane, X, Y must hence satisfy the conditions in (3.2). $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ implies that, since (G, K) is a rank one symmetric pair, we may assume $Y_{\mathfrak{p}} = 0$ without loss of generality. Hence $X \in \mathfrak{p} \oplus \mathfrak{m}, Y \in \mathfrak{m}$, since $X_{\mathfrak{g}_2} = Y_{\mathfrak{g}_2} = 0$. Now $0 = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]$ if and only if $0 = [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]$. But Theorem 2.1(i) tells us that the bi-invariant metric on $Spin(7)$ induces positive curvature on $Spin(7)/G_2 = S^7$, so there are no independent commuting vectors in $\mathfrak{m}$. Then, without loss of generality, $X \in \mathfrak{p}, Y \in \mathfrak{m}$. □

Thus we have

$$X = \left(\begin{array}{c|c} 0 & w \\ \hline -w^t & 0 \end{array}\right) \in \mathfrak{p}, \quad Y = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (v_{ij}) \end{array}\right) \in \mathfrak{m},$$

and $[X, Y] = 0$, where $w = (w_1, \dots, w_7) \in \mathbb{R}^7$ and $(v_{ij}) = (v_{ij} \mid 2 \le i, j \le 8)$.

Again applying the conditions in (3.2) we see that $\check{\sigma}_1$ has zero curvature if and only if $[(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}}, (\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}}] = 0$, since

$$[\mathrm{Ad}_{A^t} X, \mathrm{Ad}_{A^t} Y] = \mathrm{Ad}_{A^t}([X, Y]) = 0.$$

But (G, K) is a rank one symmetric pair and thus $\check{\sigma}_1$ has zero curvature if and only if $(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}}, (\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}}$ are linearly dependent. Since an element of $\mathfrak{p}$ is determined by its first row, for $A = (a_{ij}) \in SO(8)$ we have

$$(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} \sim \sum_{k,\ell=1}^{8} a_{k1} x_{k\ell} a_{\ell j} = \sum_{\ell=2}^{8} (a_{11} a_{\ell j} - a_{\ell 1} a_{1j}) w_{\ell-1}, \quad j = 2, \dots, 8,$$

and

$$(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} \sim \sum_{k,\ell=1}^{8} a_{k1} y_{k\ell} a_{\ell j} = \sum_{k,\ell=2}^{8} a_{k1} v_{k\ell} a_{\ell j}, \quad j = 2, \dots, 8.$$

If we assume that

$$A = \begin{pmatrix} R(\theta) & \\ & I_{6\times 6} \end{pmatrix} \in SO(8), \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z},$$

then

$$(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} \sim (0, w_1, w_2 \cos\theta, w_3 \cos\theta, w_4 \cos\theta, w_5 \cos\theta, w_6 \cos\theta, w_7 \cos\theta)$$

and

$$(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} \sim (0, 0, v_1 \sin\theta, v_2 \sin\theta, v_3 \sin\theta, v_4 \sin\theta, v_5 \sin\theta, v_6 \sin\theta).$$

Now $(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}}, (\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}}$ are linearly dependent if and only if either

$$(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} = 0 \quad \text{or} \quad (\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} = 0 \quad \text{or} \quad (\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} = s(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}},$$

for some $s \in \mathbb{R} - \{0\}$.

Suppose $(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} = 0$. Then $w_1 = 0$ and $w_j \cos\theta = 0$, $j = 2, \dots, 7$. But $\theta \neq \frac{n\pi}{2}$, hence $w_j = 0$ for all j and so $X = X_{\mathfrak{p}} = 0$. Thus $(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} = 0$ is impossible.

Suppose now that $(\mathrm{Ad}_{A^t} X)_{\mathfrak{p}} = s(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}}$, some $s \in \mathbb{R} - \{0\}$. Then $w_1 = 0$ and $w_j = s\ v_{j-1} \tan\theta$, $2 \leq j \leq 7$. However, $[X, Y] = 0$ implies that

$$\sum_{j=2}^{7} w_j v_{j-1} = 0.$$

Since $s \neq 0$ and $\theta \neq \frac{n\pi}{2}$, we have a contradiction.

Finally consider $(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} = 0$. $\theta \neq \frac{n\pi}{2}$ implies that $v_j = 0$, $1 \leq j \leq 6$, and so

$$Y = Y_{\mathfrak{m}} = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 & v_7 &&&& \\ && -v_7 & 0 &&&& \\ &&&& 0 & v_7 && \\ &&&& -v_7 & 0 && \\ &&&&&& 0 & -v_7 \\ &&&&&& v_7 & 0 \end{pmatrix}.$$

Recall that $\langle Y, \mathrm{Ad}_A \Theta\rangle_0 = 0$ for all $\Theta \in \mathfrak{s}$. But $\mathrm{Ad}_A \Theta = \Theta$ since A commutes with S^1. Hence we must have $v_7 = 0$, i.e. $Y = 0$. Therefore $(\mathrm{Ad}_{A^t} Y)_{\mathfrak{p}} = 0$ is impossible.

We have shown that there are no horizontal zero-curvature planes at (A, I), and therefore have proved that the image of (A, I) in $G/\!\!/U$ is a point of positive curvature. We have proved the curvature part of Theorem A(ii).

Since extending the U action to an action by $SO(3) \times G_2$ increases the number of conditions which must be satisfied in order for a zero-curvature plane to be horizontal, Theorem A(iii) follows immediately.

5. Generalised Eschenburg spaces

Consider the rank one symmetric pairs $(G, K) = (U(n+1), U(1)U(n))$ and $(K, H) = (U(1)U(n), U(1)U(1)U(n-1))$ where $n \geq 2$,

$$\begin{array}{ccc} K & \hookrightarrow & G \\ (z, A) & \longmapsto & \begin{pmatrix} z & \\ & A \end{pmatrix}, \quad z \in U(1), A \in U(n), \end{array}$$

and

$$\begin{array}{ccc} H & \hookrightarrow & K \\ (z, w, B) & \longmapsto & \left(z, \begin{pmatrix} w & \\ & B \end{pmatrix}\right), \quad z, w \in U(1), B \in U(n-1). \end{array}$$

Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H respectively. Let $\langle X, Y\rangle_0 = -\operatorname{Re} \operatorname{tr}(XY)$ be a bi-invariant metric on G. With respect to $\langle\ ,\ \rangle_0$ we thus

have the orthogonal decompositions $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$, where

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & -\bar{x}^t \\ \hline x & 0 \end{array} \right) \,\middle|\, x = \begin{pmatrix} x_2 \\ \vdots \\ x_{n+1} \end{pmatrix} \in \mathbb{C}^n \right\} \quad \text{and}$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c|c} 0 & & \\ \hline & 0 & -\bar{y}^t \\ \hline & y & 0 \end{array} \right) \,\middle|\, y = \begin{pmatrix} y_3 \\ \vdots \\ y_{n+1} \end{pmatrix} \in \mathbb{C}^{n-1} \right\}.$$

As in Examples (a) and (b) we define a left-invariant, right K-invariant metric $\langle\ ,\ \rangle_1$ on G by

$$\langle X, Y \rangle_1 = \langle X, \Phi_1(Y) \rangle_0, \tag{5.1}$$

where $\Phi_1(Y) = Y_{\mathfrak{p}} + \lambda_1 Y_{\mathfrak{k}}$, $\lambda_1 \in (0,1)$, and a left-invariant, right H-invariant metric $\langle\ ,\ \rangle_2$ on G via

$$\langle X, Y \rangle_2 = \langle X, \Psi(Y) \rangle_1 = \langle X, \Phi_2(Y) \rangle_0 \tag{5.2}$$

where $\Phi_2(Y) = Y_{\mathfrak{p}} + \lambda_1 Y_{\mathfrak{m}} + \lambda_1\lambda_2 Y_{\mathfrak{h}}$, $\lambda_2 \in (0,1)$, and $\Psi(Y) = \Phi_1^{-1}\Phi_2(Y) = Y_{\mathfrak{p}} + Y_{\mathfrak{m}} + \lambda_2 Y_{\mathfrak{h}}$.

Equip $G \times G$ with the left-invariant, right $(K \times H)$-invariant product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_2$.

Consider the subgroup $L_{p,q} \subset K \times H$ defined by

$$L_{p,q} = \{(\mathrm{diag}(z^{p_1}, \ldots, z^{p_{n+1}}), \mathrm{diag}(z^{q_1}, z^{q_2}, A)) \mid z \in S^1, A \in U(n-1)\},$$

where $p_1, \ldots, p_{n+1}, q_1, q_2 \in \mathbb{Z}$. $L_{p,q}$ acts on G via

$$\begin{aligned} G &\longrightarrow G \\ B &\longmapsto \mathrm{diag}(z^{p_1}, \ldots, z^{p_{n+1}}) B\, \mathrm{diag}(\bar{z}^{q_1}, \bar{z}^{q_2}, A^{-1}), \end{aligned}$$

where $z \in U(1)$ and $A \in U(n-1)$. It is not difficult to show that this action is free if and only if

$$(p_{\sigma(1)} - q_1, p_{\sigma(2)} - q_2) = 1 \quad \text{for all} \quad \sigma \in S_{n+1}. \tag{5.3}$$

We denote the resulting biquotients $G/\!\!/L_{p,q}$ by $E^{4n-1}_{p,q}$ and remark that $n = 2$ gives the usual Eschenburg spaces (see [E1]).

Recall the canonical diffeomorphism

$$E^{4n-1}_{p,q} = G/\!\!/L_{p,q} \cong \Delta G \backslash G \times G / L_{p,q}$$

given in (1.8). Now, since $L_{p,q} \subset K \times H$, there is a metric on $E^{4n-1}_{p,q}$ induced from the product metric on $G \times G$.

Theorem 5.1. *The biquotient $E^{4n-1}_{p,q} = U(n+1)/\!\!/L_{p,q}$ admits a metric with quasi-positive curvature whenever*

$$p_1 \neq p_2 \quad \text{and} \quad p_1 + p_2 \notin \{2q_1, 2q_2, q_1 + q_2\}.$$

Proof. We must find a point in $E^{4n-1}_{p,q} = \Delta G\backslash G\times G/L_{p,q}$ at which all 2-planes have positive curvature. As we mentioned in Section 1, this is equivalent to locating a point in $G \times G$ at which there are no horizontal zero-curvature planes since we have equipped $G \times G$ with the product metric $\langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_2$.

From (1.9) it is easy to show that the horizontal subspace at a point $(A, I) \in G \times G$ is given by

$$\mathcal{H}_A = \{(-\Phi_1^{-1}(\mathrm{Ad}_{A^*} W), \Phi_2^{-1}(W)) \mid W_{\mathfrak{u}(n-1)} = 0, \langle W, \mathrm{Ad}_A P - Q\rangle_0 = 0\}, \tag{5.4}$$

where $A^* = \bar{A}^t$, $P = \mathrm{diag}(ip_1, \ldots, ip_{n+1})$, $Q = \mathrm{diag}(iq_1, iq_2, 0, \ldots, 0)$, and $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n-1)$ as before.

In order to simplify the computations to follow we fix

$$A = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \hline & & I_{n-1} \end{array}\right)$$

for the remainder of the proof, where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix.

Before we proceed to a discussion of horizontal zero-curvature planes at (A, I) we prove the following useful lemma:

Lemma 5.2. *Suppose $p_1 + p_2 \notin \{2q_1, 2q_2, q_1 + q_2\}$. Then a vector of the form $(-\Phi_1^{-1}(\mathrm{Ad}_{A^*} W), \Phi_2^{-1}(W))$ with*

$$W = \mathrm{diag}(i, 0, \ldots, 0),\ \mathrm{diag}(0, i, \ldots, 0), \quad \text{or} \ \mathrm{diag}(i, i, 0, \ldots, 0),$$

cannot be horizontal at (A, I).

Proof. Consider $V = \mathrm{diag}(i\theta, i\varphi, 0, \ldots, 0)$. From (5.4) we see that the vector $(-\Phi_1^{-1}(\mathrm{Ad}_{A^*} V), \Phi_2^{-1}(V))$ is horizontal if and only if $\langle W, \mathrm{Ad}_A P - Q\rangle_0 = 0$. Since $\langle X, Y\rangle_0 = -\,\mathrm{Re}\ \mathrm{tr}(XY)$ and by our choice of A, this is equivalent to the condition

$$\theta q_1 + \varphi q_2 = \frac{1}{2}(\theta + \varphi)(p_1 + p_2).$$

The result now easily follows. □

Suppose that

$$\sigma = \mathrm{Span}\left\{\left(-\Phi_1^{-1}\left(\mathrm{Ad}_{A^*} \widetilde{X}\right), \Phi_2^{-1}\left(\widetilde{X}\right)\right), \left(-\Phi_1^{-1}\left(\mathrm{Ad}_{A^*} \widetilde{Y}\right), \Phi_2^{-1}\left(\widetilde{Y}\right)\right)\right\}$$

is a horizontal zero-curvature plane at $(A, I) \in G \times G$. From the discussion in Section 1 we know that the projections $\check{\sigma}_i$, $i = 1, 2$, onto the first and second factor must be two-dimensional zero-curvature planes with respect to $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ respectively.

Consider first

$$\check{\sigma}_2 = \mathrm{Span}\left\{\Phi_2^{-1}\left(\widetilde{X}\right), \Phi_2^{-1}\left(\widetilde{Y}\right)\right\} = \mathrm{Span}\left\{\Psi^{-1}(X), \Psi^{-1}(Y)\right\},$$

where $\Psi = \Phi_1^{-1}\Phi_2$ and $W = \Phi_1^{-1}(\widetilde{W})$. $\check{\sigma}_2$ has zero-curvature with respect to $\langle\ ,\ \rangle_2$ if and only if the equalities in (1.6) hold, namely if and only if

$$0 = [X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{m}}, Y_{\mathfrak{m}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}].$$

Since (G, K) is a rank-one symmetric pair, $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ if and only if $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are linearly dependent. Without loss of generality we may assume that $Y_{\mathfrak{p}} = 0$. Similarly, (K, H) being a rank-one symmetric pair implies that $[X_{\mathfrak{m}}, Y_{\mathfrak{m}}] = 0$ if and only if $X_{\mathfrak{m}}$ and $Y_{\mathfrak{m}}$ are linearly dependent. Without loss of generality we may assume that either $X_{\mathfrak{m}} = 0$ or $Y_{\mathfrak{m}} = 0$. Thus we have two possibilities:

$$X = X_{\mathfrak{p}} + X_{\mathfrak{m}} + X_{\mathfrak{h}} \text{ and } Y = Y_{\mathfrak{h}}; \quad \text{or} \tag{5.5}$$

$$X = X_{\mathfrak{p}} + X_{\mathfrak{h}} \text{ and } Y = Y_{\mathfrak{m}} + Y_{\mathfrak{h}}. \tag{5.6}$$

Since σ is horizontal and Φ_1 simply scales $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$ by $\lambda_1 \in (0, 1)$, then we must have $X_{\mathfrak{u}(n-1)} = Y_{\mathfrak{u}(n-1)} = 0$, where $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n-1)$. Therefore in both cases above we have

$$X_{\mathfrak{h}} = \mathrm{diag}(ia, ib, 0, \dots, 0), \quad Y_{\mathfrak{h}} = \mathrm{diag}(ic, id, 0, \dots, 0), \quad \text{some } a, b, c, d \in \mathbb{R}.$$

Clearly $[X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$. Then $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{m}}, Y_{\mathfrak{h}}] + [X_{\mathfrak{h}}, Y_{\mathfrak{m}}]$. In the case of (5.5) our zero-curvature condition is thus $0 = [X_{\mathfrak{m}}, Y_{\mathfrak{h}}]$, while for case (5.6) we have $0 = [X_{\mathfrak{h}}, Y_{\mathfrak{m}}]$.

Consider general vectors $Z = \mathrm{diag}(i\alpha, i\beta, 0, \dots, 0) \in \mathfrak{h}$ and $W \in \mathfrak{m}$. Then $0 = [Z, W]$ if and only if either $\beta = 0$ or $W = 0$. Applying this to case (5.5) we find (after rescaling)

$$X = X_{\mathfrak{p}} + X_{\mathfrak{m}} + X_{\mathfrak{h}} \text{ and } Y = \mathrm{diag}(i, 0, \dots, 0); \quad \text{or} \tag{5.7}$$

$$X = X_{\mathfrak{p}} + X_{\mathfrak{h}} \text{ and } Y = Y_{\mathfrak{h}}. \tag{5.8}$$

On the other hand, case (5.6) yields the added possibility

$$X = \left(\begin{array}{c|c} i\alpha & -\bar{x}^t \\ \hline x & 0 \end{array}\right) \in \mathfrak{p} \oplus \mathfrak{h} \text{ and } Y = Y_{\mathfrak{m}} + Y_{\mathfrak{h}}. \tag{5.9}$$

By Lemma 5.2 we can immediately rule out case (5.7) and concentrate on cases (5.8) and (5.9).

The only zero-curvature condition remaining to us is $[X, Y] = 0$. Since $Y_{\mathfrak{p}} = 0$, this is equivalent to $[X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = 0$. Consider the general vectors

$$U = \left(\begin{array}{c|c} 0 & -\bar{u}^t \\ \hline u & 0 \end{array}\right) \in \mathfrak{p} \text{ and } V = \left(\begin{array}{c|c|c} i\gamma & & \\ \hline & i\delta & -\bar{v}^t \\ \hline & v & 0 \end{array}\right) \in \mathfrak{k}$$

where $u = (u_2, \dots, u_{n+1}) \in \mathbb{C}^n$, $v = (v_3, \dots, v_{n+1}) \in \mathbb{C}^{n-1}$, and $\gamma, \delta \in \mathbb{R}$. Then

$$\begin{aligned} &[U, V] = 0 \\ \Longleftrightarrow\ & i(\gamma - \delta)u_2 + \sum_{j=3}^{n+1} u_j \bar{v}_j = 0 \text{ and } i\gamma u_j = u_2 v_j,\ 3 \leq j \leq n+1 \end{aligned} \tag{5.10}$$

Suppose $u_2 = 0$. Then (5.10) becomes $\gamma u_j = 0$, $3 \leq j \leq n+1$, and $\sum_{j=3}^{n+1} u_j \bar{v}_j = 0$. This is satisfied if and only if either

$$u_j = 0 \ \text{ for all } \ j = 2, \dots, n+1, \quad \text{i.e. } \ X_{\mathfrak{p}} = 0; \quad \text{or} \tag{5.11}$$

$$u_2 = 0 \ \text{ and } \ \gamma = 0 \ \text{ and } \ \sum_{j=3}^{n+1} u_j \bar{v}_j = 0. \tag{5.12}$$

On the other hand, if we assume $u_2 \neq 0$ then (5.10) becomes

$$\begin{aligned} &u_2 \neq 0 \ \text{ and } \ v_j = \gamma \left(\frac{i\bar{u}_2}{|u_2|^2} \right) u_j, j = 3, \dots, n+1, \quad \text{and} \\ &\delta = \frac{\gamma}{|u_2|^2} \left(|u_2|^2 - \sum_{j=3}^{n+1} |u_j|^2 \right). \end{aligned} \tag{5.13}$$

Now, if we apply conditions (5.11), (5.12) and (5.13) to case (5.8) we arrive at (after rescaling where appropriate)

$$X = \operatorname{diag}(ia, ib, 0, \dots, 0) \text{ and } Y = \operatorname{diag}(ic, id, 0, \dots, 0); \quad \text{or} \tag{5.14}$$

$$X \in \mathfrak{p} \oplus \mathfrak{h} \ \text{ and } \ Y = \operatorname{diag}(0, i, 0, \dots, 0); \ \text{or} \tag{5.15}$$

$$X \in \mathfrak{p} \oplus \mathfrak{h} \ \text{ and } \ Y = \operatorname{diag}(i, i, 0, \dots, 0). \tag{5.16}$$

Since X and Y must span a two-plane, it is clear that $\operatorname{diag}(i, 0, \dots, 0)$ must lie in the plane spanned by the X and Y given in (5.14). Hence we may apply Lemma 5.2 to conclude that there are no horizontal zero-curvature planes $\sigma \subset \mathfrak{g} \oplus \mathfrak{g}$ described by case (5.8).

For case (5.9) conditions (5.11), (5.12) and (5.13) imply that X and Y must have one of the following forms (after rescaling):

$$X = \operatorname{diag}(i, 0, \dots, 0) \ \text{ and } \ Y \in \mathfrak{k}; \tag{5.17}$$

or

$$X = \left(\begin{array}{c|cccc} i\alpha & 0 & -\bar{x}_3 & \cdots & -\bar{x}_{n+1} \\ \hline 0 & & & & \\ x_3 & & & & \\ \vdots & & & 0 & \\ x_{n+1} & & & & \end{array} \right), \quad Y = \left(\begin{array}{c|c|c} 0 & & \\ \hline & i\beta & -\bar{y}^t \\ \hline & & \\ & y & 0 \\ & & \end{array} \right) \tag{5.18}$$

where $(x_3, \dots, x_{n+1}) \neq 0 \in \mathbb{C}^{n-1}$ and $\sum_{j=3}^{n+1} x_j \bar{y}_j = 0$; or, finally,

$$X = \left(\begin{array}{c|c} i\alpha & -\bar{x}^t \\ \hline x & 0 \end{array}\right) \quad \text{and} \quad Y = \left(\begin{array}{c|c|c} i & & \\ \cline{2-3} & i\beta & -\bar{y}^t \\ \cline{2-3} & y & 0 \end{array}\right) \tag{5.19}$$

where $x_2 \neq 0$, $\beta = |x_2|^2 - \sum_{j=3}^{n+1} |x_j|^2$, and $y_j = \left(\frac{i\bar{x}_2}{|x_2|^2}\right) x_j$ for $j = 3, \dots, n+1$.

Applying Lemma 5.2 once again eliminates case (5.17). Therefore, in order to complete the proof we may restrict our attention to horizontal zero-curvature planes for which X and Y are of one of the forms given in (5.18) and (5.19).

Without loss of generality we may assume that the vectors $\Psi^{-1}(X)$ and $\Psi^{-1}(Y)$ spanning $\check{\sigma}_2$ are orthogonal. By (5.2) and since $Y \in \mathfrak{k}$ this is equivalent to $\langle X_{\mathfrak{h}}, Y_{\mathfrak{h}} \rangle_0 = 0$, where we recall that $\langle V, W \rangle_0 = -\operatorname{Re} \operatorname{tr}(VW)$. This is trivially true for X and Y of the form (5.18), but for (5.19) we get orthogonality if and only if $\alpha = 0$. Hence we may rewrite (5.19) as

$$X = \left(\begin{array}{c|c} 0 & -\bar{x}^t \\ \hline x & 0 \end{array}\right) \quad \text{and} \quad Y = \left(\begin{array}{c|c|c} i & & \\ \cline{2-3} & i\beta & -\bar{y}^t \\ \cline{2-3} & y & 0 \end{array}\right) \tag{5.20}$$

where $x_2 \neq 0$, $\beta = |x_2|^2 - \sum_{j=3}^{n+1} |x_j|^2$, and $y_j = \left(\frac{i\bar{x}_2}{|x_2|^2}\right) x_j$ for $j = 3, \dots, n+1$.

We now turn our attention to the projection $\check{\sigma}_1$ of σ onto the first factor. Recall that

$$\begin{aligned} \check{\sigma}_1 &= \operatorname{Span}\left\{\Phi_1^{-1}\left(\operatorname{Ad}_{A^*} \widetilde{X}\right), \Phi_1^{-1}\left(\operatorname{Ad}_{A^*} \widetilde{Y}\right)\right\} \\ &= \operatorname{Span}\left\{\Phi_1^{-1}\left(\operatorname{Ad}_{A^*}(\Phi_1 X)\right), \Phi_1^{-1}\left(\operatorname{Ad}_{A^*}(\Phi_1 Y)\right)\right\}. \end{aligned}$$

(1.3) provides us with conditions for $\check{\sigma}_1$ to have zero-curvature with respect to $\langle\ ,\ \rangle_1$ but, since we have already assumed that $\check{\sigma}_2$ has zero-curvature and since (G, K) is a rank-one symmetric pair, the conditions reduce to

$$\left[\left(\operatorname{Ad}_{A^*}(\Phi_1 X)\right)_{\mathfrak{p}}, \left(\operatorname{Ad}_{A^*}(\Phi_1 Y)\right)_{\mathfrak{p}}\right] = 0.$$

That is, the $\mathfrak{p}$ components of $\operatorname{Ad}_{A^*}(\Phi_1 X)_{\mathfrak{p}}$ and $\operatorname{Ad}_{A^*}(\Phi_1 Y)_{\mathfrak{p}}$ must be linearly dependent. There are three possible cases:

$$\begin{aligned} &\left(\operatorname{Ad}_{A^*}(\Phi_1 X)\right)_{\mathfrak{p}} = 0; \quad \text{or} \\ &\left(\operatorname{Ad}_{A^*}(\Phi_1 Y)\right)_{\mathfrak{p}} = 0; \quad \text{or} \\ &\left(\operatorname{Ad}_{A^*}(\Phi_1 X)\right)_{\mathfrak{p}} = s\left(\operatorname{Ad}_{A^*}(\Phi_1 Y)\right)_{\mathfrak{p}}, \end{aligned}$$

for some $s \in \mathbb{R} - \{0\}$.

Recall that $\Phi_1(W) = W_{\mathfrak{p}} + \lambda_1 W_{\mathfrak{k}}$, $\lambda_1 \in (0,1)$, that we have chosen

$$A = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \hline & & \\ & & I_{n-1} \\ & & \end{array}\right)$$

and that

$$\mathrm{Ad}_{A^*} V = \sum_{k,\ell=1}^{n+1} \bar{a}_{ki} a_{\ell j} v_{k\ell},$$

where $V = (v_{ij}) \in \mathfrak{g}$. Then, since $\mathfrak{p}$ is completely determined by the first row of vectors in $\mathfrak{g}$, we may abuse notation to write

$$(\mathrm{Ad}_{A^*} V)_{\mathfrak{p}} = \left(\sum_{k,\ell=1}^{n+1} \bar{a}_{k1} a_{\ell j} v_{k\ell} \;\middle|\; j = 2, \ldots, n+1 \right).$$

Let us consider the two possible pairs (X, Y) separately, beginning with (5.18). Here we have

$$(\mathrm{Ad}_{A^*}(\Phi_1 X))_{\mathfrak{p}} = \left(-\frac{1}{2}\, i\lambda_1 \alpha, -\frac{1}{\sqrt{2}}\, \bar{x}_3, \ldots, -\frac{1}{\sqrt{2}}\, \bar{x}_{n+1} \right), \quad (5.21)$$

$$(\mathrm{Ad}_{A^*}(\Phi_1 Y))_{\mathfrak{p}} = \left(\frac{1}{2}\, i\lambda_1 \beta, -\frac{1}{\sqrt{2}}\, \bar{y}_3, \ldots, -\frac{1}{\sqrt{2}}\, \bar{y}_{n+1} \right). \quad (5.22)$$

If $(\mathrm{Ad}_{A^*}(\Phi_1 X))_{\mathfrak{p}} = 0$ then $X = 0$, which is contradiction since σ is two-dimensional. Similarly, $(\mathrm{Ad}_{A^*}(\Phi_1 Y))_{\mathfrak{p}} = 0$ gives a contradiction. On the other hand, if $\left(\mathrm{Ad}_{A^*}(\Phi_1 X)\right)_{\mathfrak{p}} = s\left(\mathrm{Ad}_{A^*}(\Phi_1 Y)\right)_{\mathfrak{p}}$ for some non-zero $s \in \mathbb{R}$ then we find $x_j = s y_j$, $j = 3, \ldots, n+1$. However, since $\sum_{j=3}^{n+1} x_j \bar{y}_j = 0$, this implies that $x_j = y_j = 0$ for all $j = 3, \ldots, n+1$. But x_j cannot all be zero for vectors of type (5.18), and so we have a contradiction. Therefore there are no horizontal zero-curvature planes $\sigma \subset \mathfrak{g} \oplus \mathfrak{g}$ described by (5.18).

We now consider the pair (X, Y) given in (5.20). We have

$$(\mathrm{Ad}_{A^*}(\Phi_1 X))_{\mathfrak{p}} = \left(-\frac{1}{2}(\bar{x}_2 + x_2), -\frac{1}{\sqrt{2}}\, \bar{x}_3, \ldots, -\frac{1}{\sqrt{2}}\, \bar{x}_{n+1} \right), \quad (5.23)$$

$$(\mathrm{Ad}_{A^*}(\Phi_1 Y))_{\mathfrak{p}} = \left(-\frac{1}{2}\lambda_1(1-\beta), -\frac{1}{\sqrt{2}}\, \bar{y}_3, \ldots, -\frac{1}{\sqrt{2}}\, \bar{y}_{n+1} \right). \quad (5.24)$$

If $(\mathrm{Ad}_{A^*}(\Phi_1 X))_{\mathfrak{p}} = 0$ then $y_j = 0$ for all $j = 3, \ldots, n+1$, since $y_j = \left(\frac{i\bar{x}_2}{|x_2|^2}\right) x_j$, $j = 3, \ldots, n+1$. The only non-zero entry in X is x_2 and we may assume without loss of generality that $|x_2| = 1$. Thus $\beta = 1$ and $Y = \mathrm{diag}(i, i, 0, \ldots, 0)$ since $\beta = |x_2|^2 - \sum_{j=3}^{n+1} |x_j|^2$. By Lemma 5.2 this is impossible.

If $(\mathrm{Ad}_{A^*}(\Phi_1 Y))_{\mathfrak{p}} = 0$ then we may once again assume without loss of generality that $Y = \mathrm{diag}(i, i, 0, \ldots, 0)$ and Lemma 5.2 gives a contradiction.

Finally we examine the situation $(\mathrm{Ad}_{A^*}(\Phi_1 X))_{\mathfrak{p}} = s\,(\mathrm{Ad}_{A^*}(\Phi_1 Y))_{\mathfrak{p}}$ for some non-zero $s \in \mathbb{R}$. Then $x_j = sy_j$ implies that $y_j = s\left(\frac{i\bar{x}_2}{|x_2|^2}\right) y_j$, $j = 3, \dots, n+1$, since $y_j = \left(\frac{i\bar{x}_2}{|x_2|^2}\right) x_j$, $j = 3, \dots, n+1$. We have already shown that the y_j cannot all be zero. Therefore we find that $s\left(\frac{i\bar{x}_2}{|x_2|^2}\right) = 1$. This in turn implies that $x_2 = is \in i\mathbb{R}$.

Now $\left(-\Phi_1^{-1}(\mathrm{Ad}_{A^*}(\Phi_1 X)), \Phi_2^{-1}(\Phi_1 X)\right)$ is a horizontal vector if and only if $\langle X, \mathrm{Ad}_A P - Q\rangle_0 = 0$, i.e. if and only if

$$\begin{aligned} 0 &= \sum_{\ell=1}^{n+1} \mathrm{Im}\left(a_{1\ell} \sum_{k=2}^{n+1} \bar{a}_{k\ell} x_k\right) p_\ell \\ &= \mathrm{Im}\,(a_{11}\bar{a}_{21}x_2)\,p_1 + \mathrm{Im}\,(a_{12}\bar{a}_{22}x_2)\,p_2 \\ &= \frac{1}{2}\,\mathrm{Im}\,(x_2)(p_1 - p_2), \end{aligned}$$

where again we recall that $\langle V, W\rangle_0 = -\mathrm{Re}\ \mathrm{tr}(VW)$. By hypothesis $p_1 \neq p_2$ and so we must have $x_2 = \mathrm{Im}\,(x_2) = 0$, which contradicts the assumption that $x_2 \neq 0$. □

Remark 5.3. As was discussed in Section 1, a permutation of the integers $p_1, \dots, p_{n+1}$ induces a diffeomorphism $E^{4n-1}_{p,q} \longrightarrow E^{4n-1}_{p,q}$. Therefore the proof of Theorem 5.1 is sufficient to establish Theorem A(i).

Remark 5.4. Theorem A(i) is certainly not optimal since, for example, the spaces defined by $p = (0, \dots, 0)$ and $q = (-1, 1)$ satisfy neither hypothesis of the theorem, but are known to admit metrics with almost positive curvature constructed in a similar manner [Wi]. It is the author's suspicion that all generalised Eschenburg spaces admit quasi-positive curvature, as in the case $n = 2$ [Ke2].

6. Topology of M^{13}, N^{11}

We turn now to the topological assertions of Theorem A regarding the biquotients $M^{13} = SO(8)/\!\!/(S^1 \times G_2)$ and $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$, namely that they have the same cohomology rings but are not homeomorphic to $\mathbb{C}P^3 \times S^7$ and $S^4 \times S^7$ respectively.

Theorem 6.1. *The biquotients M^{13} and N^{11} have the same cohomology rings as $\mathbb{C}P^3 \times S^7$ and $S^4 \times S^7$ respectively. In particular M^{13} and N^{11} are not manifolds known to admit positive curvature.*

Proof. Consider any circle bundle

$$S^1 \longrightarrow S^7 \times S^7 \xrightarrow{p} M^{13}.$$

The long exact homotopy sequence for fibre bundles implies that $\pi_1(M^{13}) = 0$. Hence there is a Gysin sequence for the bundle $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$,

$$.. \longrightarrow H^{i-2}(M;\mathbb{Z}) \xrightarrow{\smile e} H^i(M;\mathbb{Z}) \xrightarrow{p^*} H^i(S^7 \times S^7;\mathbb{Z}) \longrightarrow H^{i-1}(M;\mathbb{Z}) \longrightarrow ..$$

Recall that

$$H^j(S^7 \times S^7; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 14; \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } j = 7; \\ 0, & \text{otherwise.} \end{cases}$$

Since we have a circle bundle $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$, there is an isomorphism $H^0(M; \mathbb{Z}) \cong H^0(S^7 \times S^7) = \mathbb{Z}$, and the Euler class $e \in H^2(M; \mathbb{Z})$.

The Gysin sequence gives groups $H^j(M; \mathbb{Z}) = \mathbb{Z}$, $j = 0, 2, 4, 6$, and $H^j(M; \mathbb{Z}) = 0$, $j = 1, 3, 5$. By Poincaré Duality and the Universal Coefficient Theorem we thus have

$$H^j(M^{13}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 2, 4, 6, 7, 9, 11, 13; \\ 0, & \text{if } j = 1, 3, 5, 8, 10, 12. \end{cases}$$

Hence, looking at the Serre spectral sequence for a fibration $S^1 \longrightarrow S^7 \times S^7 \longrightarrow M^{13}$, we see that M^{13} has the same cohomology ring as $\mathbb{C}P^3 \times S^7$, namely $H^*(M; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^4, \beta^2 \rangle$, where $\alpha \in H^2(M)$ and $\beta \in H^7(M)$.

The analogous Gysin sequence computation for $S^3 \longrightarrow S^7 \times S^7 \longrightarrow N^{11}$ yields

$$H^j(N^{11}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, 4, 7, 11; \\ 0, & \text{if } j = 1, 2, 3, 5, 6, 8, 9, 10, \end{cases}$$

and the Euler class $e \in H^4(N^{11}; \mathbb{Z})$. Looking at the Serre spectral sequence for a fibration $S^3 \longrightarrow S^7 \times S^7 \longrightarrow N^{11}$ we find that N^{11} has cohomology ring structure $H^*(N^{11}; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^2, \beta^2 \rangle$, , where $\alpha \in H^4(N^{11})$ and $\beta \in H^7(N^{11})$, and so N^{11} has the same cohomology as $S^4 \times S^7$. □

Before we continue we establish an easy lemma which will prove useful in the topological computations to follow.

Lemma 6.2. *Consider a triple (r_1, r_2, r_3) such that $\sum r_i = 0$. Let $\sigma_i(r)$ and $\sigma_i(r^2)$ denote the i^{th} elementary symmetric polynomials in r_1, r_2, r_3 and r_1^2, r_2^2, r_3^2 respectively. Then $\sigma_1(r^2) = -2\sigma_2(r)$ and $\sigma_2(r^2) = \sigma_2(r)^2$.*

Proof. Since $\sigma_1(r) = \sum r_i = 0$ we have

$$\begin{aligned} 0 &= \sigma_1(r)^2 \\ &= (r_1^2 + r_2^2 + r_3^2) + 2(r_1 r_2 + r_1 r_3 + r_2 r_3) \\ &= \sigma_1(r^2) + 2\sigma_2(r) \end{aligned}$$

as desired. On the other hand

$$\begin{aligned} \sigma_2(r)^2 - \sigma_2(r^2) &= (r_1 r_2 + r_1 r_3 + r_2 r_3)^2 - (r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2) \\ &= 2(r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2) \\ &= 2 r_1 r_2 r_3 (r_1 + r_2 + r_3) \\ &= 0. \end{aligned}$$

□

In [E1] (pages *vii* and 139), Eschenburg provides a beautiful diagram which explicitly describes the embedding of the root system G_2 into B_3. Recall that B_3 is the root system corresponding to the Lie algebra $\mathfrak{so}(7)$ and is given by

$$B_3 = \{\pm t_i \mid 1 \le i \le 3\} \cup \{\pm(t_i \pm t_j) \mid 1 \le i < j \le 3\}.$$

The root system G_2 lies on a hypersurface in Span $\{B_3\}$ and is given by

$$G_2 = \{\pm s_i \mid 1 \le i \le 3\} \cup \{\pm(s_i - s_j) \mid 1 \le i < j \le 3\},$$

where $s_i = \frac{1}{3}(2t_i - t_j - t_k)$, $\{i,j,k\} = \{1,2,3\}$. Notice that $\sum s_i = 0$ and that $s_i - s_j = t_i - t_j \in B_3$. Furthermore, s_i is the projection of $t_i \in B_3$ and $-(t_j + t_k) \in B_3$ onto the hypersurface containing G_2.

Since the Lie group G_2 is simply connected and has no centre, we see that the inclusions

$$\exp^{-1}(I) = \text{integral lattice of } G_2 \subset \text{root lattice of } G_2 \subset \text{weight lattice of } G_2$$

are in fact equalities. Therefore, by our above discussion of the roots of G_2, the integral and weight lattices of G_2 are spanned by $\{s_i \mid 1 \le i \le 3\}$, $\sum s_i = 0$. Thus by an abuse of notation we may assume that $\{s_i \mid 1 \le i \le 3\}$, $\sum s_i = 0$, spans $H^1(T_{G_2};\mathbb{Z}) = \mathrm{Hom}(\Gamma,\mathbb{Z})$, where T_{G_2} is a maximal torus of G_2 and Γ is the integral lattice of G_2.

Recall that Lemma 3.2 showed that

$$M^{13} := SO(8)/\!\!/(S^1 \times G_2)$$

is a quotient of $S^7 \times S^7$ by a particular S^1 action.

Theorem 6.3. *The first Pontrjagin class of M^{13} is*

$$p_1(M^{13}) \quad = \quad 2\alpha^2$$

where α is a generator of $H^2(M^{13};\mathbb{Z}_p) = \mathbb{Z}_p$, p prime, $p \ge 3$.

Prior to providing the proof we remark that, in terms of integral cohomology, the theorem tells us only that $p_1(M^{13})$ is not divisible by any primes $p \ge 3$. Thus $p_1(M^{13}) = \pm 2^k \in \mathbb{Z} = H^4(M^{13})$, for some $k \in \mathbb{Z}$, $k \ge 0$. Since $p(M_1 \times M_2) = p(M_1) \otimes p(M_2)$, $p(S^n) = 1$, and $p(\mathbb{C}P^n) = (1+\beta^2)^{n+1}$, where β is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$, we find that $p_1(\mathbb{C}P^3 \times S^7) = 4(\beta \otimes 1)^2$. Therefore we are unable to distinguish M^{13} and $\mathbb{C}P^3 \times S^7$ using the theorem.

Proof of Theorem 6.3. We follow the techniques developed in [BH], [E2] and [Si] (see also [FZ]).

Let $G = SO(8)$. Define the inclusions

$$\begin{array}{rcl} f_q : H := S^1_q & \hookrightarrow & G \\ R(u) & \longmapsto & \begin{pmatrix} R(q_1 u) & & & \\ & R(q_2 u) & & \\ & & R(q_3 u) & \\ & & & R(q_4 u) \end{pmatrix} \end{array}$$

where $q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}^4$, and

$$g : K := G_2 \hookrightarrow G$$

given by the embedding of G_2 into $SO(7) \subset SO(8)$. For a general Lie group L, let E_L denote a contractible space on which L acts freely, and denote the classifying space E_L/L by B_L. For the sake of notation we denote a product of Lie groups $L_1 \times L_2$ by $L_1 L_2$.

Consider the following commutative diagram of fibrations

$$\begin{tikzcd}
G \times E_{GG} \arrow[r] \arrow[d] & G \times E_{GG} \arrow[d] \\
G \times_{HK} E_{GG} \arrow[r, "\varphi_G"] \arrow[d, "(\varphi_H, \varphi_K)"'] & G \times_{GG} E_{GG} = B_{\Delta G} \arrow[d, "B_\Delta"] \\
B_H \times B_K \arrow[r, "(B_{f_q}, B_g)"'] & B_G \times B_G
\end{tikzcd} \tag{6.1}$$

where φ_G, φ_K, and φ_H are the respective classifying maps, and $\Delta : G \longrightarrow GG$ denotes the diagonal embedding. Now, since projection onto the first factor in each case is a homotopy equivalence, we have $G \simeq G \times E_{GG}$ and $H \backslash G/K \simeq G \times_{HK} E_{GG}$. Thus, up to homotopy, we can consider the diagram as

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \arrow[d] \\
H \backslash G/K \arrow[r] \arrow[d] & B_{\Delta G} \arrow[d, "B_\Delta"] \\
B_H \times B_K \arrow[r, "(B_{f_q}, B_g)"'] & B_G \times B_G
\end{tikzcd}$$

Recall that $SO(8)$ and G_2 have torsion in their cohomology for coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$ (see [MT]). Therefore, using $\mathbb{Z}_p$ coefficients with $p \geq 3$ and prime, we have

$$\begin{aligned}
H^*(G; \mathbb{Z}_p) &= \Lambda(y_1, y_2, y_3, y_4), \quad y_1 \in H^3, y_2, y_4 \in H^7, y_3 \in H^{11}, \\
H^*(K; \mathbb{Z}_p) &= \Lambda(x_1, x_2), \quad x_1 \in H^3, x_2 \in H^{11}, \\
H^*(H; \mathbb{Z}_p) &= \Lambda(u), \quad u \in H^1.
\end{aligned}$$

Hence

$$\begin{aligned}
H^*(B_G; \mathbb{Z}_p) &= \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4], \quad \bar{y}_1 \in H^4, \bar{y}_2, \bar{y}_4 \in H^8, \bar{y}_3 \in H^{12} \\
H^*(B_K; \mathbb{Z}_p) &= \mathbb{Z}_p[\bar{x}_1, \bar{x}_2], \quad \bar{x}_1 \in H^4, \bar{x}_2 \in H^{12} \\
H^*(B_H; \mathbb{Z}_p) &= \mathbb{Z}_p[\bar{u}], \quad \bar{u} \in H^2,
\end{aligned}$$

where $\bar{y}_i$, $\bar{x}_j$ and $\bar{u}$ denote the transgressions of y_i, x_j and u respectively. Let T_G and T_K be the maximal tori of G and K respectively, with coordinates being given by (t_1, t_2, t_3, t_4) and (s_1, s_2, s_3), $\sum s_i = 0$, respectively. By an

abuse of notation (and our earlier discussion of the roots of G_2) we will identify t_i and s_j with the elements $t_i \in H^1(T_G)$ and $s_j \in H^1(T_K)$. The corresponding transgressions are $\bar{t}_i \in H^2(B_{T_G})$ and $\bar{s}_j \in H^2(B_{T_K})$. Since G and K do not have any torsion in their cohomologies we have

$$\begin{aligned} H^*(B_G) &= H^*(B_{T_G})^{W_G} = \mathbb{Z}_p[\bar{t}_1, \dots, \bar{t}_4]^{W_G} \quad \text{and} \\ H^*(B_K) &= H^*(B_{T_K})^{W_K} = \mathbb{Z}_p[\bar{s}_1, \bar{s}_2, \bar{s}_3]^{W_K} \end{aligned}$$

where W_L denotes the Weyl group of L.

W_G acts on $H^*(B_{T_G})$ via permutations in $\bar{t}_i$ and an even number of sign changes. Therefore a basis for $H^*(B_{T_G})^{W_G}$ is given by elementary symmetric polynomials $\sigma_i(\bar{t}^2) := \sigma_i(\bar{t}_1^2, \dots, \bar{t}_4^2)$, $i = 1, 2, 3$, and $\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$. Hence we choose $\bar{y}_i = \sigma_i(\bar{t}^2)$, $i = 1, 2, 3$, and $\bar{y}_4 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$.

$W_{G_2} = W_K$ is the dihedral group of order twelve. Its action on the root system G_2 is by rotations of $\frac{\pi}{3}$ and by reflections through the horizontal axis. Therefore, given our description of the root system of G_2 above, W_K acts on $H^*(B_{T_K})$ via permutations in $\bar{s}_i$ and a simultaneous sign change of all $\bar{s}_i$. Thus elements of $H^*(B_{T_K})$ which are invariant under W_K are given by sums and products of the elementary symmetric polynomials $\sigma_2(\bar{s}) := \sigma_2(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ and $\sigma_i(\bar{s}^2) := \sigma_i(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $i = 1, 2, 3$. However, since $\sum s_i = 0$, Lemma 6.2 shows that a basis for $H^*(B_{T_K})^{W_K}$ is given by the symmetric polynomials $\sigma_i(\bar{s}^2)$, $i = 1, 3$. Thus we identify $\bar{x}_1 = \sigma_1(\bar{s}^2)$ and $\bar{x}_2 = \sigma_3(\bar{s}^2)$.

Therefore we have

$$H^*(B_G; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4], \tag{6.2}$$

$$H^*(B_K; \mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{s}^2), \sigma_3(\bar{s}^2)]. \tag{6.3}$$

Let $h : L_1 \longrightarrow L_2$ be a homomorphism of Lie groups. Then the commutative diagram

$$\begin{tikzcd} L_1 \arrow[r, "h"] & L_2 \\ T_{L_1} \arrow[u, hook] \arrow[r, "h"'] & T_{L_2} \arrow[u, hook] \end{tikzcd}$$

induces a commutative diagram of classifying spaces

$$\begin{tikzcd} B_{L_1} \arrow[r, "B_h"] & B_{L_2} \\ B_{T_{L_1}} \arrow[u] \arrow[r, "B_h"'] & B_{T_{L_2}} \arrow[u] \end{tikzcd} \tag{6.4}$$

which in turn induces the commutative diagram

$$\begin{tikzcd} H^*(B_{L_1}) \arrow[d] & H^*(B_{L_2}) \arrow[l, "(B_h)^*"'] \arrow[d] \\ H^*(B_{T_{L_1}}) & H^*(B_{T_{L_2}}) \arrow[l, "(B_h)^*"] \end{tikzcd} \tag{6.5}$$

Recall that

$$\begin{aligned} H^*(B_{GG}) = H^*(B_G) \otimes H^*(B_G) &= H^*(B_{T_G})^{W_G} \otimes H^*(B_{T_G})^{W_G} \\ &= \mathbb{Z}_p[\bar{t}_1, \dots, \bar{t}_4]^{W_G} \otimes \mathbb{Z}_p[\bar{t}_1, \dots, \bar{t}_4]^{W_G} \\ &= \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4] \otimes \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4], \end{aligned}$$

where $\mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4] = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4]$ as before. Consider the diagonal embedding $\Delta : G \hookrightarrow GG$. In coordinates $\Delta|_{T_G}$ is given by $t_i \longmapsto (t_i, t_i)$, $i = 1, \dots, 4$. We have commutative diagrams as in (6.4) and (6.5). Now

$$\begin{aligned} \Delta^* : H^1(T_G) \otimes H^1(T_G) &\longrightarrow H^1(T_G) \\ t_i \otimes 1 &\longmapsto t_i \\ 1 \otimes t_i &\longmapsto t_i, \end{aligned}$$

which in turn implies

$$\begin{aligned} (B_\Delta)^* : H^2(B_{T_G}) \otimes H^2(B_{T_G}) &\longrightarrow H^2(B_{T_G}) \\ \bar{t}_i \otimes 1 &\longmapsto \bar{t}_i \\ 1 \otimes \bar{t}_i &\longmapsto \bar{t}_i. \end{aligned}$$

Therefore,

$$\begin{aligned} (B_\Delta)^* : H^*(B_{GG}) &\longrightarrow H^*(B_G) \\ \bar{y}_i \otimes 1 &\longmapsto \bar{y}_i \\ 1 \otimes \bar{y}_i &\longmapsto \bar{y}_i. \end{aligned}$$

Since the diagram (6.1) is commutative we see that

$$\begin{aligned} \varphi_G^*(\bar{y}_i) &= \varphi_G^*((B_\Delta)^*(\bar{y}_i \otimes 1)) = \varphi_H^*((B_{f_q})^*(\bar{y}_i)), \quad \text{and} \\ \varphi_G^*(\bar{y}_i) &= \varphi_G^*((B_\Delta)^*(1 \otimes \bar{y}_i)) = \varphi_K^*((B_g)^*(\bar{y}_i)). \end{aligned}$$

Consider now $f_q : H := S^1_q \hookrightarrow G$ as above. In coordinates f_q is given by

$$u \longmapsto (q_1 u, \dots, q_4 u).$$

We get commutative diagrams as in (6.4) and (6.5). Now

$$\begin{aligned} (f_q)^* : H^1(T_G) &\longrightarrow H^1(S^1_q) \\ t_i &\longmapsto q_i u, \end{aligned}$$

which implies that

$$\begin{array}{rcl}(B_{f_q})^* : H^2(B_{T_G}) & \longrightarrow & H^2(B_{S^1_q}) \\ \bar{t}_i & \longmapsto & q_i \bar{u}.\end{array}$$

Therefore, letting $q^2 := (q_1^2, \dots, q_4^2)$, we have

$$\begin{array}{rcl}(B_{f_q})^* : H^*(B_G) & \longrightarrow & H^*(B_{S^1_q}) \\ \bar{y}_i & \longmapsto & \sigma_i(q^2)\bar{u}^{2i}, \quad i = 1, 2, 3, \\ \bar{y}_4 & \longmapsto & \sigma_4(q)\bar{u}^4.\end{array} \tag{6.6}$$

On the other hand, now consider $g : K := G_2 \hookrightarrow G \subset SO(8)$ as above. In particular, $g|_{T_K} : T_K \longrightarrow T_G$, and examining T_K as in (2.3) we see that in coordinates $g|_{T_K}$ is given by $(s_1, s_2, s_3) \longmapsto (0, s_1, s_2, -s_3)$, $\sum s_i = 0$. Again we get commutative diagrams as in (6.4) and (6.5). Now

$$\begin{array}{rcl}(g|_{T_K})^* : H^1(T_G) & \longrightarrow & H^1(T_K) \\ t_1 & \longmapsto & 0, \\ t_i & \longmapsto & s_{i-1}, \quad i = 2, 3, \\ t_4 & \longmapsto & -s_3\end{array}$$

and hence

$$\begin{array}{rcl}(B_{g|_{T_K}})^* : H^2(B_{T_G}) & \longrightarrow & H^2(B_{T_K}) \\ \bar{t}_1 & \longmapsto & 0, \\ \bar{t}_i & \longmapsto & \bar{s}_{i-1}, \quad i = 2, 3, \\ \bar{t}_4 & \longmapsto & -\bar{s}_3\end{array}$$

Therefore we have

$$\begin{array}{rcl}(B_g)^* : H^*(B_G) & \longrightarrow & H^*(B_K) \\ \bar{y}_1 & \longmapsto & \sigma_1(\bar{s}^2) = \bar{x}_1, \\ \bar{y}_2 & \longmapsto & \sigma_2(\bar{s}^2), \\ \bar{y}_3 & \longmapsto & \sigma_3(\bar{s}^2) = \bar{x}_2, \\ \bar{y}_4 & \longmapsto & 0.\end{array} \tag{6.7}$$

Thus $(B_g)^*(\bar{y}_1) = \bar{x}_1$ and $(B_g)^*(\bar{y}_3) = \bar{x}_2$.

We are now in a position to compute the Pontrjagin class of $H\backslash G/K$, and in particular p_1. Let τ be the tangent bundle of $H\backslash G/K$. In [Si] the following vector bundles over $H\backslash G/K$ were introduced. Let $\alpha_H := (G/K) \times_H \mathfrak{h}$, where H acts on G/K on the left, and on $\mathfrak{h}$ via Ad_H. Let $\alpha_K := (H\backslash G) \times_K \mathfrak{k}$, where K acts on $H\backslash G$ on the right, and on $\mathfrak{k}$ via Ad_K. Finally, let $\alpha_G := ((H\backslash G) \times (G/K)) \times_G \mathfrak{g}$, where G acts on $(H\backslash G) \times (G/K)$ via $(Hg_1, g_2K) \star g = (Hg_1 g, g^{-1} g_2 K)$, and on $\mathfrak{g}$ via Ad_G. Since $H \times K$ acts freely on G we have

$$\tau \oplus \alpha_H \oplus \alpha_K = \alpha_G.$$

Recall from [BH] that the Pontrjagin class of a homogeneous vector bundle $\alpha_L = P \times_L V$ associated to the L-principal bundle $P \longrightarrow B := P/L$ is given by

$$p(\alpha_L) = 1 + p_1(\alpha_L) + p_2(\alpha_L) + \cdots = \varphi_L^*(a), \quad a := \prod_{\alpha_i \in \Delta_L^+} (1 + \bar{\alpha}_i^2),$$

where Δ_L^+ is the set of positive weights of the representation of L on V, and $\varphi_L : B \longrightarrow B_L$ is the classifying map of the L-principal bundle. We have identified $\alpha_i \in H^1(T_L) \cong H^2(B_{T_L})$, and hence $a \in H^*(B_{T_L})^{W_L} \cong H^*(B_L)$

In our case the vector bundles α_H, α_K, α_G are associated to a principal bundle and the weights are the roots of the corresponding Lie group.

Since $H = S^1$ we have $p(\alpha_H) = 1$, and since, if V, W are vector bundles over some manifold M, $p(V \oplus W) = p(V) \smile p(W)$, we have

$$p(\tau)p(\alpha_K) = p(\alpha_G).$$

By our discussion above and since inverses are well-defined in the polynomial algebra $H^*(B_K)$ it follows that

$$p(\tau) = \varphi_G^*(a)\varphi_K^*(b^{-1}),$$

where $a := \prod_{\alpha_i \in \Delta_G^+}(1 + \bar{\alpha}_i^2)$ and $b := \prod_{\beta_j \in \Delta_K^+}(1 + \bar{\beta}_j^2)$. In particular, note that

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_K) \\ &= \varphi_G^* \left(\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 \right) - \varphi_K^* \left(\sum_{\beta_j \in \Delta_K^+} \bar{\beta}_j^2 \right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \le i < j \le 4$. Hence

$$\begin{aligned} \sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 &= \sum_{1 \le i < j \le 4} \left((\bar{t}_i - \bar{t}_j)^2 + (\bar{t}_i + \bar{t}_j)^2 \right) \\ &= 2 \sum_{1 \le i < j \le 4} (\bar{t}_i^2 + \bar{t}_j^2) \\ &= 6 \sum_{i=1}^{4} \bar{t}_i^2 \\ &= 6\bar{y}_1. \end{aligned}$$

Now $\varphi_G^*(\bar{y}_1) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2)$. Therefore we may conclude $p_1(\alpha_G) = 6\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$.

From our earlier description of the roots of G_2, the positive roots of $K = G_2$ are

$$s_1,\ s_2,\ -s_3,\ s_1 - s_3,\ s_2 - s_1,\ s_2 - s_3,$$

where $\sum s_i = 0$. Then

$$\begin{aligned}
\sum_{\beta_j \in \Delta_K^+} \bar{\beta}_j^2 &= \bar{s}_1^2 + \bar{s}_2^2 + \bar{s}_3^2 + (\bar{s}_1 - \bar{s}_3)^2 + (\bar{s}_2 - \bar{s}_1)^2 + (\bar{s}_2 - \bar{s}_3)^2 \\
&= \sigma_1(\bar{s}^2) + (\bar{s}_1^2 + \bar{s}_3^2 - 2\bar{s}_1\bar{s}_3) + (\bar{s}_1^2 + \bar{s}_2^2 - 2\bar{s}_1\bar{s}_2) + (\bar{s}_2^2 + \bar{s}_3^2 - 2\bar{s}_2\bar{s}_3) \\
&= 3\sigma_1(\bar{s}^2) - 2\sigma_2(\bar{s}) \\
&= 4\sigma_1(\bar{s}^2) \qquad \text{by Lemma 6.2} \\
&= 4\bar{x}_1
\end{aligned}$$

Thus, since

$$\varphi_K^*(\bar{x}_1) = \varphi_K^*((B_g)^*(\bar{y}_1)) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2),$$

we have $p_1(\alpha_K) = 4\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$. Hence

$$\begin{aligned}
p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_K) \\
&= 2\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)
\end{aligned}$$

In our case we have $q = (0,0,0,1)$. Therefore $p_1(M^{13}) = p_1(\tau) = 2\varphi_H^*(\bar{u}^2)$.

Consider the Serre spectral sequence for the fibration $G \longrightarrow H\backslash G/K \longrightarrow B_{HK}$. Notice that $\bar{u} \in H^2(B_H) = H^2(B_{HK}) = E_2^{2,0}$ will survive until E_∞ since $H^*(G)$ contains no elements of degree 1. Recall that the classifying map φ_H^* is the edge homomorphism

$$\varphi_H^* : H^i(B_{HK}) = E_2^{i,0} \twoheadrightarrow E_\infty^{i,0} \hookrightarrow H^i(H\backslash G/K).$$

Therefore, given that M^{13} has the same cohomology ring as $\mathbb{C}P^3 \times S^7$, $\varphi_H^*(\bar{u})$ is mapped to a non-zero element, i.e. a generator, of $H^2(H\backslash G/K; \mathbb{Z}_p) = \mathbb{Z}_p$ and hence $\varphi_H^*(\bar{u}^2) \neq 0$. □

Remark 6.4. This proof in fact yields a more general statement. Given any simply connected biquotient $S^1\backslash SO(8)/G_2$ (i.e. the sum of the weights $q_1, \ldots, q_4$ of the embedding $SO(2) \hookrightarrow SO(8)$ is odd; see Lemma 3.2) it is easy to check that $\sigma_1(q^2)$ is always odd. Thus, if $\sigma_1(q^2) \neq 1$ (namely when the embedding has weights different from a permutation of $(0,0,0,1)$), then the first Pontrjagin class will be zero mod p for some odd primes p. This allows us to distinguish each of the corresponding biquotients $S^1\backslash SO(8)/G_2$ from $\mathbb{C}P^3 \times S^7$, and often from each other. For example, the family of actions with weights $(1,1,1,k)$, $(k,3) = 1$, described in (3.5) give simply connected biquotients whenever $k = 2\ell$, and have first Pontrjagin class $p_1(G/\!\!/U) = 2(4\ell^2 + 3)\varphi_H^*(\bar{u}^2) \in H^4(G/\!\!/U) = \mathbb{Z}_p$.

Let us now turn our attention to N^{11} where we have better luck than in Theorem 6.3.

Theorem 6.5. *The manifold $N^{11} = SO(8)/\!\!/(SO(3) \times G_2)$ has first Pontrjagin class*

$$p_1(N^{11}) = \alpha,$$

where α is a generator of $H^4(N^{11}; \mathbb{Z}_p) = \mathbb{Z}_p$, p prime, $p \geq 3$.

Proof. We let G and K be as above, $H = SO(3)$, and retain the notation and techniques used in the proof of Theorem 6.3.

Again using $\mathbb{Z}_p$ coefficients with $p \geq 3$ and prime, we have

$$\begin{aligned} H^*(G;\mathbb{Z}_p) &= \Lambda(y_1, y_2, y_3, y_4), \quad y_1 \in H^3, y_2, y_4 \in H^7, y_3 \in H^{11}, \\ H^*(K;\mathbb{Z}_p) &= \Lambda(x_1, x_2), \quad x_1 \in H^3, x_2 \in H^{11}, \\ H^*(H;\mathbb{Z}_p) &= \Lambda(w), \quad w \in H^3. \end{aligned}$$

Hence

$$\begin{aligned} H^*(B_G;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4], \quad \bar{y}_1 \in H^4, \bar{y}_2, \bar{y}_4 \in H^8, \bar{y}_3 \in H^{12} \\ H^*(B_K;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{x}_1, \bar{x}_2], \quad \bar{x}_1 \in H^4, \bar{x}_2 \in H^{12} \\ H^*(B_H;\mathbb{Z}_p) &= \mathbb{Z}_p[\bar{w}], \quad \bar{w} \in H^2, \end{aligned}$$

where $\bar{y}_i$, $\bar{x}_j$ and $\bar{w}$ denote the transgressions of y_i, x_j and w respectively.

Let T_H be the maximal tori of H with coordinate u. By an abuse of notation we will identify u with the element $u \in H^1(T_H)$. Hence $\bar{u} \in H^2(B_{T_H})$. Since G, K and H do not have any torsion in their cohomologies we have

$$\begin{aligned} H^*(B_G) &= H^*(B_{T_G})^{W_G} = \mathbb{Z}_p[\bar{t}_1, \ldots, \bar{t}_4]^{W_G}, \\ H^*(B_K) &= H^*(B_{T_K})^{W_K} = \mathbb{Z}_p[\bar{s}_1, \bar{s}_2, \bar{s}_3]^{W_K}, \quad \text{and} \\ H^*(B_H) &= H^*(B_{T_H})^{W_H} = \mathbb{Z}_p[\bar{u}]^{W_H}, \end{aligned}$$

where W_L denotes the Weyl group of L.

W_G acts on $H^*(B_{T_G})$ via permutations in $\bar{t}_i$ and an even number of sign changes. Therefore a basis for $H^*(B_{T_G})^{W_G}$ is given by elementary symmetric polynomials $\sigma_i(\bar{t}^2) := \sigma_i(\bar{t}_1^2, \ldots, \bar{t}_4^2)$, $i = 1, 2, 3$, and $\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$. Hence we choose $\bar{y}_i = \sigma_i(\bar{t}^2)$, $i = 1, 2, 3$, and $\bar{y}_4 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$.

Similarly, since W_K is the dihedral group of order twelve and acts on $H^*(B_{T_K})$ via permutations in $\bar{s}_i$ and a simultaneous sign change, a basis for $H^*(B_{T_G})^{W_K}$ is given by the symmetric polynomials $\sigma_i(\bar{s}^2) := \sigma_i(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $i = 1, 3$. Thus we identify $\bar{x}_1 = \sigma_1(\bar{s}^2)$ and $\bar{x}_2 = \sigma_3(\bar{s}^2)$.

W_H acts on $H^*(B_{T_H})$ via a sign change. Hence a basis for $H^*(B_{T_H})^{W_H}$ is given by $\bar{u}^2$.

Therefore we have

$$H^*(B_G;\mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{t}^2), \sigma_2(\bar{t}^2), \sigma_3(\bar{t}^2), \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4], \tag{6.8}$$

$$H^*(B_K;\mathbb{Z}_p) = \mathbb{Z}_p[\sigma_1(\bar{s}^2), \sigma_3(\bar{s}^2)], \tag{6.9}$$

$$H^*(B_H;\mathbb{Z}_p) = \mathbb{Z}_p[\bar{u}^2]. \tag{6.10}$$

Recall from the proof of Theorem 6.3 that

$$\varphi_K^*((B_g)^*(\bar{y}_i)) = \varphi_G^*(\bar{y}_i) = \varphi_H^*((B_{f_q})^*(\bar{y}_i)).$$

We have already computed $(B_g)^*$ and $(B_{f_q})^*$, in particular for $q = (0, 0, 0, 1)$.

We may now compute the Pontrjagin classes of N^{11}, and in particular p_1. Given the vector bundles α_G, α_K and α_H defined as before,

$$\begin{aligned} p_1(N^{11}) &= p_1(\alpha_G) - p_1(\alpha_K) - p_1(\alpha_H) \\ &= \varphi_G^*\left(\sum_{\alpha_i \in \Delta_G^+} \alpha_i^2\right) - \varphi_K^*\left(\sum_{\beta_j \in \Delta_K^+} \beta_j^2\right) - \varphi_H^*\left(\sum_{\gamma_j \in \Delta_H^+} \gamma_j^2\right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \leq i < j \leq 4$. We have already shown that $\sum_{\alpha_i} \bar{\alpha}_i^2 = 6\bar{y}_1$. Now $\varphi_G^*(\bar{y}_1) = \varphi_H^*((B_{f_q})^*(\bar{y}_1)) = \sigma_1(q^2)\varphi_H^*(\bar{u}^2)$. Hence $p_1(\alpha_G) = 6\sigma_1(q^2)\varphi_H^*(\bar{u}^2) \in H^4(H\backslash G/K)$.

The positive roots of $K = G_2$ are

$$s_1,\ s_2,\ -s_3,\ s_1 - s_3,\ s_2 - s_1,\ s_2 - s_3,$$

where $\sum s_i = 0$. Then $\sum_{\beta_j} \bar{\beta}_j^2 = 4\bar{x}_1$ and $p_1(\alpha_K) = 4\sigma_1(q^2)\varphi_H^*(\bar{u}^2)$ as before.

There is only one positive root for $SO(3)$, namely u. Therefore $\sum_{\gamma_j} \bar{\gamma}_j^2 = \bar{u}^2$ and, since $q = (0,0,0,1)$,

$$\begin{aligned} p_1(N^{11}) &= p_1(\alpha_G) - p_1(\alpha_K) - p_1(\alpha_H) \\ &= \varphi_H^*(\bar{u}^2) \in H^4(N^{11}). \end{aligned}$$

It remains to show that $\varphi_H^*(\bar{u}^2) \neq 0$. Consider the Serre spectral sequence for the fibration $G \longrightarrow N^{11} \longrightarrow B_{HK}$. By our earlier computations of $(B_{f_q})^*$ and $(B_g)^*$ the generator $y_1 \in H^3(G) = E_2^{0,3} = E_3^{0,3}$ gets mapped under d_3 to

$$d_3(y_1) = (B_{f_q})^*(\bar{y}_1) - (B_g)^*(\bar{y}_1) = \bar{u}^2 - \sigma_1(\bar{s}^2) \in E_3^{4,0} = E_2^{4,0} = H^4(B_{HK}).$$

But $H^4(B_{HK})$ has generators $\bar{u}^2$ and $\sigma_1(\bar{s}^2)$, both of which are mapped to zero by d_3. Thus, in E_4 the $E_4^{4,0}$ term is a $\mathbb{Z}_p$ generated by $\bar{u}^2$ and survives to E_∞. Now the edge homomorphism $(\varphi_H^*, \varphi_K^*)$ is given by

$$(\varphi_H^*, \varphi_K^*) : H^i(B_{HK}) = E_2^{i,0} \twoheadrightarrow E_\infty^{i,0} \hookrightarrow H^i(H\backslash G/K).$$

Therefore $(\varphi_H^*, \varphi_K^*)(\bar{u}^2) = \varphi_H^*(\bar{u}^2) \neq 0$. □

Recall that we have computed p_1 using $\mathbb{Z}_p$ coefficients, $p \geq 3$. Therefore, as in Theorem 6.3, we have proved only that, for integral coefficients, $p_1(N^{11}) = \pm 2^k \in \mathbb{Z}$, for some $k \in \mathbb{Z}$, $k \geq 0$. However, recall that all Pontrjagin classes for spheres are trivial and that integral Pontrjagin classes are homeomorphism invariants. Hence

Corollary 6.6. *N^{11} is not homeomorphic to $S^4 \times S^7$.*

Remark 6.7. Since H^8 and H^{12} are trivial for each of the manifolds M^{13} and N^{11}, we have in fact computed their total Pontrjagin classes $p = 1 + p_1$ in $\mathbb{Z}_p$ coefficients.

We return now to the problem of distinguishing M^{13} and $\mathbb{C}P^3 \times S^7$. We will do this by "hot-wiring" the technique for computing Pontrjagin classes in the absence of torsion in the cohomology groups so that we can compute the integral Pontrjagin class of M^{13}.

Before we begin we establish some topological statements which will be used in the proof of Theorem 6.10. From now on we will always assume that our cohomology groups have integral coefficients, and by spectral sequence we will always mean Serre spectral sequence.

Proposition 6.8. *B_{G_2}, the classifying space of G_2, has low-dimensional integral cohomology groups $H^1 = H^2 = H^3 = H^5 = 0$ and $H^4 = \mathbb{Z}$ with generator $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$, where $\sigma_1(\bar{s}^2) := \sigma_1(\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2)$, $\sum \bar{s}_i = 0$, and $\bar{s}_i \in H^2(B_{T_{G_2}})$, $i = 1, 2, 3$, are the transgressions of the elements $s_i \in H^1(T_{G_2})$, $i = 1, 2, 3$, which span the integral lattice of G_2.*

Proof. Consider the universal bundle $G_2 \longrightarrow E_{G_2} \longrightarrow B_{G_2}$ where E_{G_2} is contractible. From [Wh], page 360, we know that $H^j(G_2) = 0$, $j = 1, 2, 4, 5$, and $H^3(G_2) = \mathbb{Z}$. Let x be a generator of $H^3(G_2)$. Since E_{G_2} is contractible all entries in the spectral sequence for the fibration $G_2 \longrightarrow E_{G_2} \longrightarrow B_{G_2}$ must get killed off. Since $d_4 : E_4^{0,3} \longrightarrow E_4^{4,0}$ is the only possible non-trivial differential with domain $E_4^{0,3}$ it must map $x \in H^3(G_2)$ to a generator $\bar{x}$ of $H^4(B_{G_2})$, and so $H^4(B_{G_2}) = \mathbb{Z}$. Similarly it is clear from the spectral sequence that $H^j(B_{G_2}) = 0$ for $j = 1, 2, 3, 5$.

Now consider the fibration $S^6 = G_2/SU(3) \longrightarrow B_{SU(3)} \longrightarrow B_{G_2}$. The spectral sequence associated to this fibration shows that $\bar{x} \in E_2^{4,0} = H^4(B_{G_2})$ survives to E_∞. Thus, since there are no other non-zero entries on the corresponding diagonal in E_∞, we see that $H^4(B_{G_2}) = H^4(B_{SU(3)})$. Recall that $H^*(B_{SU(3)})$ is a polynomial algebra generated by the elementary symmetric polynomials $\sigma_i(\bar{s}) = \sigma_i(\bar{s}_1, \bar{s}_2, \bar{s}_3)$, $i = 2, 3$, in the transgressions $\bar{s}_j$ of $s_j \in H^1(T_{SU(3)})$, $j = 1, 2, 3$, where the s_j span the integral lattice of $SU(3)$. Note that $\sum s_j = 0$, $T_{G_2} = T_{SU(3)}$ and $\deg(\sigma_i(\bar{s})) = 2i$. Therefore $H^4(B_{G_2})$ is generated by $\sigma_2(\bar{s})$. However, by Lemma 6.2 we see that $\sigma_2(\bar{s}) = -\frac{1}{2}\sigma_1(\bar{s}^2)$. We set $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$. □

Proposition 6.9. *The low-dimensional integral cohomology groups of the manifold $SO(8)/G_2 = (S^7 \times S^7)/\mathbb{Z}_2$ are $H^j(SO(8)/G_2) = H^j(\mathbb{R}P^7)$, $0 \leq j \leq 6$.*

Proof. Consider the spectral sequence for the fibration

$$\mathbb{R}P^7 = SO(7)/G_2 \longrightarrow SO(8)/G_2 \longrightarrow SO(8)/SO(7) = S^7.$$

Recall that

$$H^j(\mathbb{R}P^7) = \begin{cases} \mathbb{Z} & \text{if } j = 0, 7 \\ \mathbb{Z}_2 & \text{if } j = 2, 4, 6 \\ 0 & \text{if } j = 1, 3, 5. \end{cases}$$

It is clear that each $E_2^{0,j} = H^j(\mathbb{R}P^7)$, $j \leq 5$, survives to E_∞. For $E_2^{0,6} = H^6(\mathbb{R}P^7) = \mathbb{Z}_2$ notice that there are no non-trivial homomorphisms $\mathbb{Z}_2 \longrightarrow \mathbb{Z}$ and so the differential $d_7 : E_7^{0,6} = \mathbb{Z}_2 \longrightarrow E_7^{7,0} = \mathbb{Z}$ must be trivial. Therefore $E_2^{0,6} = H^6(\mathbb{R}P^7)$ also survives to E_∞. Since there are no other non-zero entries on the corresponding diagonals we get the desired result. $\square$

We are now in a position to complete the proof of Theorem A(ii).

Theorem 6.10. *The first integral Pontrjagin class of the biquotient $M^{13} = SO(8)/\!\!/(S^1 \times G_2)$ is given by*

$$|p_1(M^{13})| = 8y^2,$$

where y is a generator of $H^2(M^{13}) = \mathbb{Z}$.

In particular, M^{13} is not homeomorphic to $\mathbb{C}P^3 \times S^7$.

Proof. Recall that diagram (6.1) is

$$\begin{tikzcd}
G \times E_{GG} \arrow[r] \arrow[d] & G \times E_{GG} \arrow[d] \\
G \times_U E_{GG} \arrow[r, "\varphi_G"] \arrow[d, "\varphi_U"'] & G \times_{GG} E_{GG} = B_{\Delta G} \arrow[d, "B_\Delta"] \\
B_U \arrow[r, "B_\iota"'] & B_{GG}
\end{tikzcd}$$

which, up to homotopy, is the same as

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \arrow[d] \\
G/\!\!/U \arrow[r, "\varphi_G"] \arrow[d, "\varphi_U"'] & B_{\Delta G} \arrow[d, "B_\Delta"] \\
B_U \arrow[r, "B_\iota"'] & B_{GG}
\end{tikzcd}$$

where $G = SO(8)$, $U = HK = S^1 \times G_2$, and $G/\!\!/U = M^{13}$. We have altered the previous notation slightly so that $\varphi_U = (\varphi_H, \varphi_K)$ and ι is the embedding $(f_q, g) : U \hookrightarrow GG$ for $q = (0,0,0,1)$. In the proofs of Theorem 6.3 and Theorem 6.5 we followed the usual techniques of [BH], [E2] and [Si] when there is no torsion in cohomology, namely we computed B_ι and B_Δ and then used the fact that the diagram commutes in order to compute the $\mathbb{Z}_p$, $p \geq 3$, Pontrjagin class. However, since $SO(8)$ and G_2 have torsion in integral cohomology, we need to adopt a different approach in order to compute the integral Pontrjagin class. Since $H^8(M^{13}) = H^{12}(M^{13}) = 0$ we can restrict our attention to the first integral Pontrjagin class $p_1(M^{13}) \in H^4(M^{13})$. The key idea to be taken from the proofs of Theorem 6.3 and Theorem 6.5 is that we computed the first Pontrjagin classes of some vector bundles over $B_{\Delta G}$ and B_U, then pulled them back to M^{13} under the classifying maps

φ_G and φ_U respectively. As it turns out, the first Pontrjagin classes of the vector bundles over $B_{\Delta G}$ and B_U are the same in integral coefficients as in $\mathbb{Z}_p$ coefficients, $p \geq 3$. Our strategy, therefore, is to compute the maps $\varphi_U^* : H^4(B_U) \longrightarrow H^4(M^{13})$ and $\varphi_G^* : H^4(B_{\Delta G}) \longrightarrow H^4(M^{13})$ and pull back the respective first Pontrjagin classes.

As a first step in computing $\varphi_U^* : H^4(B_U) \longrightarrow H^4(M^{13})$ we notice that $H^*(U) = H^*(S^1) \otimes H^*(G_2)$ and $H^*(B_U) = H^*(B_{S^1}) \otimes H^*(B_{G_2})$ since $H^*(S^1)$ and $H^*(B_{S^1})$ are torsion-free. Therefore

$$H^j(U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle w \rangle & \text{if } j = 1 \\ \mathbb{Z} = \langle x \rangle & \text{if } j = 3 \\ 0 & \text{if } j = 2, 4, 5 \end{cases}$$

where w is a generator of $H^1(S^1)$ and x is a generator of $H^3(G_2)$, and applying Proposition 6.8

$$H^j(B_U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle \bar{w} \rangle & \text{if } j = 2 \\ \mathbb{Z} \oplus \mathbb{Z} = \langle \bar{w}^2 \rangle \oplus \langle \bar{x} \rangle & \text{if } j = 4 \\ 0 & \text{if } j = 1, 3, 5 \end{cases}$$

where $\bar{w}$ is the transgression of w resulting from the spectral sequence for the universal bundle of S^1 and generates $H^2(B_{S^1})$ (hence generates $H^*(B_{S^1}) = \mathbb{Z}[\bar{w}]$), and $\bar{x}$ is the transgression of x resulting from the spectral sequence for the universal bundle of G_2 and generates $H^4(B_{G_2})$.

Recall that $\varphi_U : G/\!\!/U \longrightarrow B_U$ is the classifying map since we have the following diagram of principal U-bundles

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \arrow[d] \\ G \times E_U \arrow[r, "\pi_2"] \arrow[d] & E_U \arrow[d] \\ G \times_U E_U \arrow[r, "\pi_2"'] & B_U \end{tikzcd}$$

where π_2 denotes projection onto the second factor and $U \longrightarrow E_U \longrightarrow B_U$ is the universal bundle. Since E_U is contractible, projection onto the first factor gives homotopy equivalences $G \times E_U \simeq G$ and $G \times_U E_U \simeq G/\!\!/U$. Then φ_U is the resulting map $G/\!\!/U \longrightarrow B_U$ and so is the classifying map. Therefore, up to homotopy, we may consider the following commutative

diagram of fibrations

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \arrow[d] \\ G \arrow[r] \arrow[d] & E_U \arrow[d] \\ G/\!\!/U \arrow[r, "\varphi_U"'] & B_U \end{tikzcd}$$

Consider first the spectral sequence for the fibration on the left. Recall that $H^*(M^{13}) = H^*(\mathbb{C}P^3 \times S^7)$. Hence

$$H^j(G/\!\!/U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z} = \langle y \rangle & \text{if } j = 2 \\ \mathbb{Z} = \langle y^2 \rangle & \text{if } j = 4 \\ 0 & \text{if } j = 1, 3, 5. \end{cases}$$

Since $G = SO(8)$ we have from [CMV] that

$$H^j(G) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ \mathbb{Z}_2 = \langle r \rangle & \text{if } j = 2 \\ \mathbb{Z} = \langle z \rangle & \text{if } j = 3 \\ \mathbb{Z}_2 = \langle r^2 \rangle & \text{if } j = 4. \end{cases}$$

Since $H^1(G) = 0$ we see that $d_2 : E_2^{0,1} = \langle w \rangle \longrightarrow E_2^{2,0} = \langle y \rangle$ must have trivial kernel, i.e. $d_2(w) = ky$ for some $k \in \mathbb{Z}$, $k \neq 0$. Then $E_3^{0,2} = \langle y \rangle / \langle ky \rangle$ survives to E_∞ and since $H^2(G) = \mathbb{Z}_2$ we must therefore have $k = \pm 2$, i.e. $d_2(w) = \pm 2y$.

On the other hand, the spectral sequence shows that on the E_4-page we have the differential $d_4 : E_4^{0,3} = \langle x \rangle \longrightarrow E_4^{0,4} = \langle y^2 \rangle / \langle 2y^2 \rangle$. However, since $H^3(G) = \mathbb{Z}$ and $H^4(G) = \mathbb{Z}_2$, we must have $d_4(x) = 0 \in \langle y^2 \rangle / \langle 2y^2 \rangle$.

Since E_U is contractible it is clear from the spectral sequence for the fibration on the right that $d_2 : E_2^{0,1} = \langle w \rangle \longrightarrow E_2^{2,0} = \langle \bar{w} \rangle$ is an isomorphism with $d_2(w) = \bar{w}$, and $d_4 : E_4^{0,3} = \langle x \rangle \longrightarrow E_4^{4,0} = \langle \bar{w}^2 \rangle \oplus \langle \bar{x} \rangle$ is given by $d_4(x) = \bar{x}$.

By naturality of the spectral sequence we thus have for the left-hand fibration that $d_2(w) = \varphi_U^*(\bar{w}) \in \langle y \rangle$ and $d_4(x) = \varphi_U^*(\bar{x}) \in \langle y^2 \rangle / \langle 2y^2 \rangle$. Therefore, since we have already shown that $d_2(w) = 2y \in \langle y \rangle$ and $d_4(x) = 0 \in \langle y^2 \rangle / \langle 2y^2 \rangle$, we find

$$\varphi_U^*(\bar{w}) = \pm 2y \in H^2(G/\!\!/U) = \langle y \rangle \quad \text{and} \tag{6.11}$$
$$\varphi_U^*(\bar{x}) = 2ky^2 \in H^4(G/\!\!/U) = \langle y^2 \rangle, \quad \text{for some } k \in \mathbb{Z}. \tag{6.12}$$

We now turn our attention to computing $\varphi_G^* : H^4(B_{\Delta G}) \longrightarrow H^4(M^{13})$. In order to show that $\varphi_G : G/\!\!/U \longrightarrow B_{\Delta G}$ is the classifying map consider the

commutative diagram of principal G-bundles

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \arrow[d] \\
GG \times_U E_{GG} \arrow[r] \arrow[d] & GG \times_{GG} E_{GG} \arrow[d] \\
(\Delta G\backslash GG) \times_U E_{GG} \arrow[r, "\varphi_G"'] & (\Delta G\backslash GG) \times_{GG} E_{GG}
\end{tikzcd}$$

Since $GG\times_{GG}E_{GG} = E_{GG}$ and $(\Delta G\backslash GG)\times_{GG}E_{GG} = G\times_{GG}E_{GG} = B_{\Delta G}$ we see that the fibration on the right-hand side is the universal bundle for G. On the left-hand side we have $(\Delta G\backslash GG) \times_U E_{GG} = G \times_U E_{GG}$, and projection onto the first factor gives homotopy equivalences $GG \times_U E_{GG} \simeq GG/U$ and $G \times_U E_{GG} \simeq G/\!\!/U$. Thus up to homotopy the diagram becomes

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \arrow[d] \\
GG/U \arrow[r] \arrow[d] & E_{GG} \arrow[d] \\
G/\!\!/U \arrow[r, "\varphi_G"'] & B_G
\end{tikzcd}$$

as desired. Recall that $H^3(G) = \langle z\rangle$. The cohomology of B_G is described in [Br] and [F], but for our purposes we need only that

$$H^j(B_G) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1, 2 \\ \mathbb{Z}_2 & \text{if } j = 3 \\ \mathbb{Z} & \text{if } j = 4 \\ \mathbb{Z}_2 & \text{if } j = 5. \end{cases}$$

Whilst proving Proposition 3.6 in [GZ2] the authors show that, since $E = E_{GG}$ is contractible, in the spectral sequence for the bundle $G \longrightarrow E \longrightarrow B_G$ the differential $d_4 : E_4^{0,3} = \langle 2z\rangle \longrightarrow E_4^{4,0} = H^4(B_G)$ is an isomorphism, i.e. $2z$ gets mapped to a generator $\bar{z}$ of $H^4(B_G) = \mathbb{Z}$. This follows from the facts that $E_2^{2,2} = \mathbb{Z}_2$ by the Universal Coefficient Theorem, and that $d_2 : E_2^{0,3} = \langle z\rangle \longrightarrow E_2^{2,2} = \mathbb{Z}_2$ must be onto.

Therefore naturality of the spectral sequence implies that $d_4(2z) = \varphi_G^*(\bar{z})$ in the spectral sequence for the left-hand fibration $G \longrightarrow GG/U \longrightarrow G/\!\!/U$, where $H^3(G) = \langle z\rangle$ and $H^4(B_G) = \langle \bar{z}\rangle$.

In order to determine the exact value of $\varphi_G^*(\bar{z}) \in H^4(G/\!\!/U)$ we need to examine the spectral sequence for the left-hand fibration. First we must compute the cohomology of GG/U in low-dimensions. Recall that $GG/U =$

$V_{8,6} \times SO(8)/G_2$, where $V_{8,6}$ is the Stiefel manifold $SO(8)/SO(2)$. From [CMV] we find that

$$H^j(V_{8,6}) = \begin{cases} \mathbb{Z} & \text{if } j = 0,2 \\ 0 & \text{if } j = 1,3,5 \\ \mathbb{Z}_2 & \text{if } j = 4. \end{cases}$$

In Proposition 6.9 we computed the low-dimensional cohomology groups of $SO(8)/G_2$. From the general Künneth formula for cohomology ([Sp], page 247) we find that

$$H^j(GG/U) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1,3 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } j = 2 \\ \mathbb{Z}_2^3 & \text{if } j = 4 \\ \mathbb{Z}_2 & \text{if } j = 5. \end{cases}$$

Since $H^4(GG/U) = \mathbb{Z}_2^3$, in the spectral sequence for $G \longrightarrow GG/U \longrightarrow G/\!\!/U$ the differential $d_2 : E_2^{0,3} = H^3(G) = \langle z \rangle \longrightarrow E_2^{2,2} = \mathbb{Z}_2$ must be trivial, i.e. $E_2^{2,2} = \mathbb{Z}_2$ must survive to E_∞. It thus follows that $E_2^{i,j} = E_3^{i,j} = E_4^{i,j}$ for $i \leq 5, j \leq 4$. Since $H^3(GG/U) = 0$ the differential $d_4 : E_4^{0,3} = \langle z \rangle \longrightarrow E_4^{4,0} = H^4(G/\!\!/U) = \langle y^2 \rangle$ must be given by $d_4(z) = ny^2$ for some non-zero $n \in \mathbb{Z}$. On the other hand, since $H^4(GG/U) = \mathbb{Z}_2^3$, $E_4^{0,4} = E_4^{2,2} = \mathbb{Z}_2$ and $E_4^{1,3} = E_4^{3,1} = 0$, the filtration for the spectral sequence shows that $n = \pm 2$, i.e. $d_4(z) = \pm 2y^2$. But we have already shown that $d_4(2z) = \varphi_G^*(\bar{z})$. Therefore

$$\varphi_G^*(\bar{z}) = \pm 4y^2 \in H^4(G/\!\!/U) = \langle y^2 \rangle.$$

Furthermore, while proving Lemma 5.4 in [GZ2] the authors show that, by considering the spectral sequences of the fibrations $SO(8)/SO(3) \longrightarrow B_{SO(3)} \longrightarrow B_{SO(8)}$ and $SO(3)/SO(2) \longrightarrow B_{SO(2)} \longrightarrow B_{SO(3)}$, we can let $\bar{z} = \sigma_1(\bar{t}^2) = \sigma_1(\bar{t}_1^2, \bar{t}_2^2, \bar{t}_3^2, \bar{t}_4^2)$, where $(t_1, \ldots, t_4)$ are the coordinates of a maximal torus T_G of G and by abuse of notation we identify $t_i \in H^1(T_G)$ with $\bar{t}_i \in H^2(B_{T_G})$ via transgression.

We are now in a position to compute the first Pontrjagin class of $M^{13} = G/\!\!/U$. Let τ be the tangent bundle of $G/\!\!/U$. Consider the following vector bundles over $G/\!\!/U$. Let $\alpha_U := G \times_U \mathfrak{u}$, where $U = S^1 \times G_2$ acts on G via the biquotient action, and on the Lie algebra $\mathfrak{u}$ of U via Ad_U. Let $\alpha_G := (GG/U) \times_G \mathfrak{g}$, where G acts on GG/U diagonally on the left and on $\mathfrak{g}$ via Ad_G. Since U acts freely on G we have, via a similar argument to that in [Si],

$$\tau \oplus \alpha_U = \alpha_G.$$

Recall that if V, W are vector bundles over some manifold M, $p(V \oplus W) = p(V) \smile p(W)$. Hence in our case

$$p(\tau)p(\alpha_U) = p(\alpha_G).$$

Recall from [BH] that, in the absence of torsion, the Pontrjagin class of a vector bundle $\alpha_L = P \times_L V$ associated to the L-principal bundle $P \longrightarrow B := P/L$ is given by

$$p(\alpha_L) = 1+p_1(\alpha_L)+p_2(\alpha_L)+\cdots = \varphi_L^*(a), \;\; a := \prod_{\alpha_i \in \Delta_L^+} (1+\bar{\alpha}_i^2) \in H^*(B_{T_L})^{W_L},$$

where Δ_L^+ is the set of positive weights of the representation of L on V, $\varphi_L : B \longrightarrow B_L$ is the classifying map of the L-principal bundle, and W_L is the Weyl group of L. Note that in this situation $H^1(T_L) \cong H^2(B_{T_L})$, and hence $a \in H^*(B_{T_L})^{W_L} \cong H^*(B_L)$

In our case, even though we have torsion in cohomology, we are fortunate in that $H^4(B_G) \cong H^4(B_{T_G})^{W_G}$ and $H^4(B_U) \cong H^4(B_{T_U})^{W_U}$ since the generators are $\bar{z} = \sigma_1(\bar{t}^2)$ and $\bar{x} = \frac{1}{2}\sigma_1(\bar{s}^2)$ respectively. Moreover the vector bundles α_U and α_G are associated to the principal bundles $U \longrightarrow G \longrightarrow G/\!\!/U$ and $G \longrightarrow GG/U \longrightarrow G/\!\!/U$ respectively, and the weights are the roots of the corresponding Lie group.

Hence we may write

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_U) \\ &= \varphi_G^*\left(\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2\right) - \varphi_U^*\left(\sum_{\beta_j \in \Delta_U^+} \bar{\beta}_j^2\right). \end{aligned}$$

The positive roots of $G = SO(8)$ are $t_i \pm t_j$, $1 \leq i < j \leq 4$. Hence, as in the proof of Theorem 6.3,

$$\sum_{\alpha_i \in \Delta_G^+} \bar{\alpha}_i^2 = 6\sum_{i=1}^4 \bar{t}_i^2 = 6\sigma_1(\bar{t}^2) = 6\bar{z}.$$

But $\varphi_G^*(\bar{z}) = \pm 4y^2$. Hence $p_1(\alpha_G) = 6\varphi_G^*(\bar{z}) = \pm 24y^2 \in H^4(G/\!\!/U)$.

From our earlier description of the roots of G_2, and since S^1 has no roots, the positive roots of U are

$$s_1, \; s_2, \; -s_3, \; s_1 - s_3, \; s_2 - s_1, \; s_2 - s_3,$$

where $\sum s_i = 0$ and $s_i = \frac{1}{3}(2t_i - t_j - t_k)$, $\{i,j,k\} = \{1,2,3\}$. Then, as in the proof of Theorem 6.3,

$$\sum_{\beta_j \in \Delta_U^+} \bar{\beta}_j^2 = 4\sigma_1(\bar{s}^2) = 8\bar{x}.$$

Thus, since $\varphi_U^*(\bar{x}) = 2ky^2$, $p_1(\alpha_U) = 8\varphi_U^*(\bar{x}) = 16ky^2 \in H^4(G/\!\!/U)$.

Therefore

$$\begin{aligned} p_1(\tau) &= p_1(\alpha_G) - p_1(\alpha_U) \\ &= 8(\pm 3 \pm 2k)y^2 \in H^4(G/\!\!/U). \end{aligned}$$

From Theorem 6.3 we know that $p_1(\tau) = p_1(G/\!\!/U)$ is divisible only by 2. Therefore $k = \pm 1$ or $k = \pm 2$ since $\pm 3 \pm 2k$ is always odd, which in turn implies $p_1(G/\!\!/U) = \pm 8y^2$ as desired. $\square$

Remark 6.11. It is tempting to suggest that $p_1(M^{13}) = -8y^2$ since we know $\varphi_G^*(\bar{z}) = \varphi_U^*(\bar{w}^2) = 4y^2$ and hence one might expect that $\varphi_U^*(\bar{x}) = 4y^2$ (as opposed to $2y^2$) purely based on the commutativity of the diagram of fibrations and the validity of the analogous statement in our $\mathbb{Z}_p$ argument.

However, in order to make this assertion one would need to compute the maps $(B_\iota)^* : H^4(B_{GG}) \longrightarrow H^4(B_U)$ and $(B_\Delta)^* : H^4(B_{GG}) \longrightarrow H^4(B_{\Delta G})$. One can easily use the Künneth formula ([Sp], page 247) to compute the low-dimensional cohomology groups of B_{GG}. Unfortunately, for example, the spectral sequence of the right-hand fibration $G \longrightarrow B_{\Delta G} \longrightarrow B_{GG}$ is rather unwieldy and so the computation of $(B_\Delta)^* : H^4(B_{GG}) \longrightarrow H^4(B_{\Delta G})$ is quite difficult.

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Mathematisches Institut, Einsteinstr. 62, 48149 Münster, Germany
E-mail address, (M. Kerin): m.kerin@math.uni-muenster.de